LOCAL NONPARAMETRIC INFERENCE ON FUNCTIONAL DATA WITH MANIFOLD DOMAIN

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PROBLEM FORMULATION

Functional data on $L^2(D) \cap \mathcal{C}^0(D)$, where $D \subset \mathbb{R}^p$.
Aim: test **locally** a functional hypothesis H_0 against H_1.

We assume that the domain D is a Riemaniann manifold of $\mathbb{R}^p$

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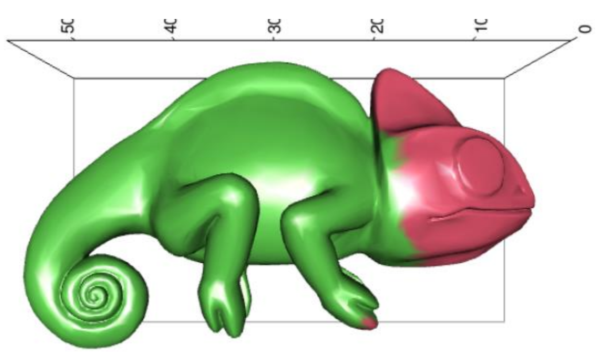
Example. Testing differences between two groups of functional data defined on a complex chamaleon-shaped manifold

$$y_i(x) = \mu_j(x) + \varepsilon_{ij}(x) \quad j = 1, 2; \; i = 1, \ldots, n_j$$

$H_0 : \mu_1(x) = \mu_2(x) \; \forall x \in D$; $\qquad H_1 : \mu_1(x) \neq \mu_2(x)$ for some $x \in D$



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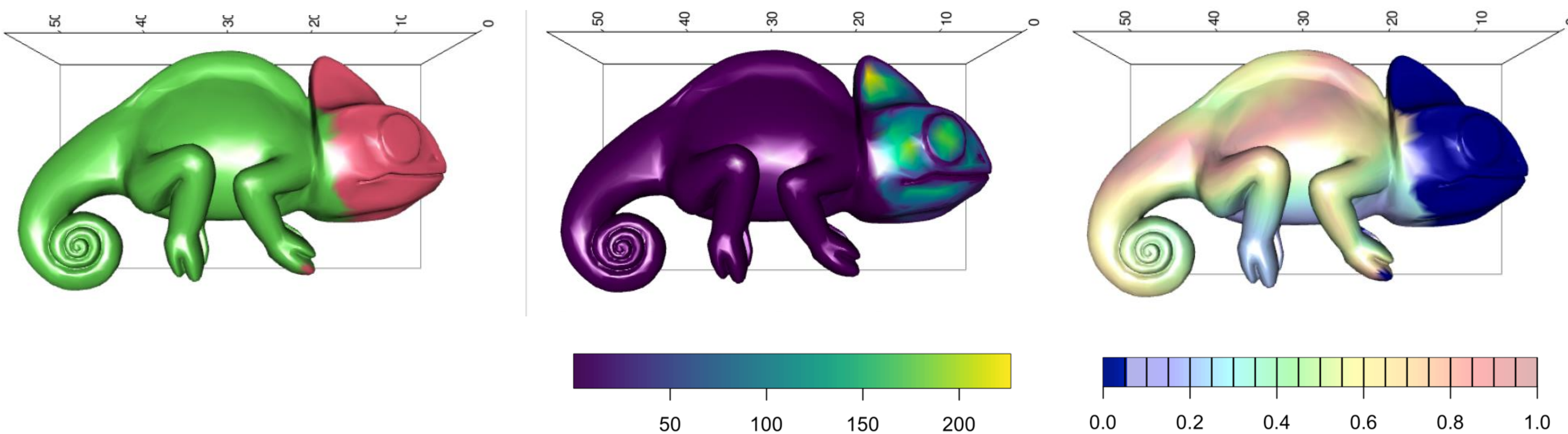
Aim: test **locally** a functional hypothesis H_0 against H_1.

True mean difference — Pointwise t-test statistic — Pointwise p-value



PROBLEM FORMULATION

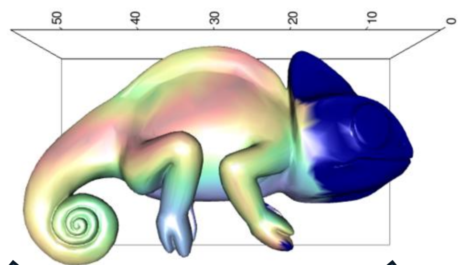
Multiplicity issue!
no control of the amount of type I errors over the domain.

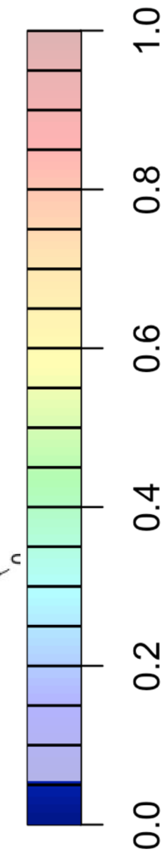
AIM

Computing an adjusted p-value function defined on D that controls an error rate defined on the whole domain.

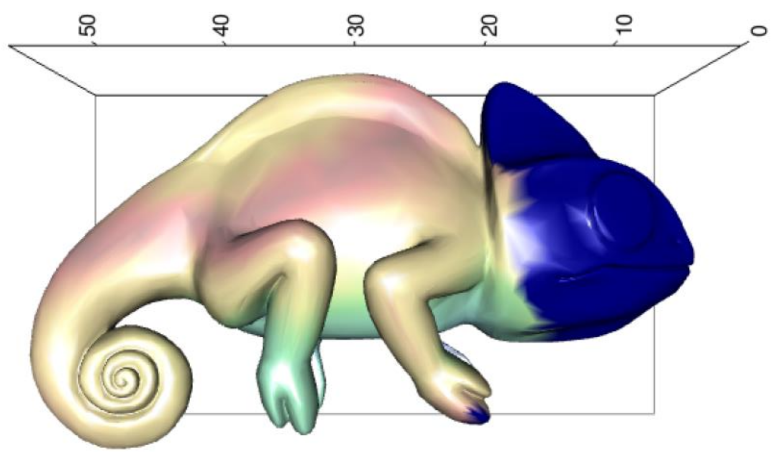
ADJUSTED P-VALUE FUNCTIONS

Pointwise p-value

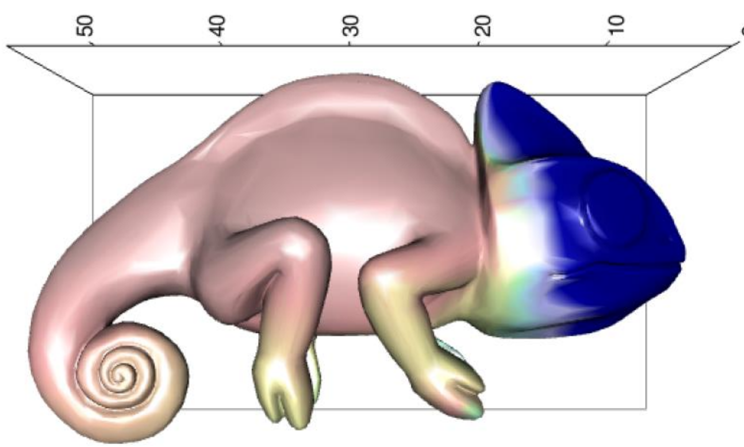




FDR adjusted p-value



Ball-wise adjusted p-value



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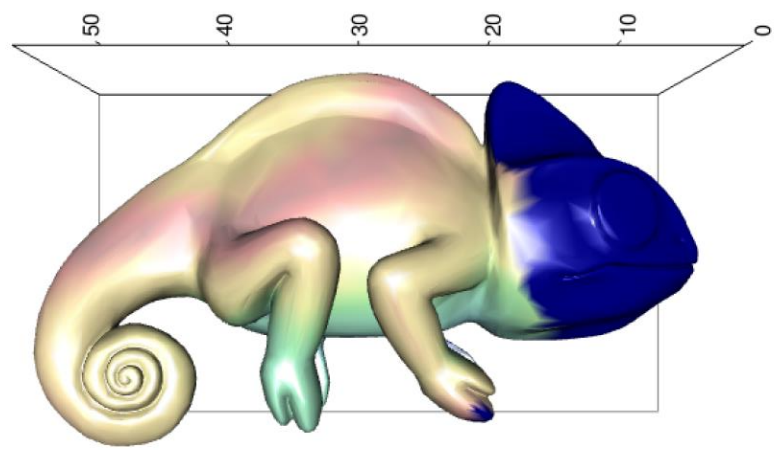
FDR-adjusted p-value function

Olsen et al 2021 TEST

$$\tilde{p}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D) \cdot s}{\mu(\{r : p(r) \leq s\})} \right\}$$



THE FUNCTIONAL FALSE DISCOVERY RATE

The FDR in multivariate statistics (Benjamini and Hochberg, 1995).

TABLE 1
Number of errors committed when testing m null hypotheses

	Declared non-significant	*Declared significant*	*Total*
True null hypotheses	**U**	**V**	m_0
Non-true null hypotheses	**T**	**S**	$m - m_0$
	$m - \mathbf{R}$	**R**	m

$$\mathrm{FDR} = \mathbb{E}\left[\frac{V}{R}\,1(R > 0)\right]$$

THE FUNCTIONAL FALSE DISCOVERY RATE

The FDR in functional data analysis.

	Declared non significant	**Declared significant**	
True null Hypotheses	U $\mu\{t \in D : H_{0t} \text{ true}, \tilde{p}(t) > \alpha\}$	V $\mu\{t \in D : H_{0t} \text{ true}, \tilde{p}(t) \leq \alpha\}$	m_0
False null Hypotheses	T $\mu\{t \in D : H_{0t} \text{ false}, \tilde{p}(t) > \alpha\}$	S $\mu\{t \in D : H_{0t} \text{ false}, \tilde{p}(t) \leq \alpha\}$	m_1
	$m - R$	R	m

$$\text{FDR} = \mathbb{E}\left[\frac{V}{R}\,1(R > 0)\right]$$

THE FUNCTIONAL BENJAMINI HOCHBERG PROCEDURE

1. Choose a suited functional test (either parametric or non-parametric).

$$H_0^{\mathcal{I}} : \mathcal{Y}_1^{\mathcal{I}} = \mathcal{Y}_2^{\mathcal{I}} \quad H_1^{\mathcal{I}} : \mathcal{Y}_1^{\mathcal{I}} \neq \mathcal{Y}_2^{\mathcal{I}}$$

2. Compute the unadjusted *p*-value function.

$$p(t) = \limsup_{\mathcal{I} \to t} p^{\mathcal{I}}$$

3. Perform functional FDR procedure:

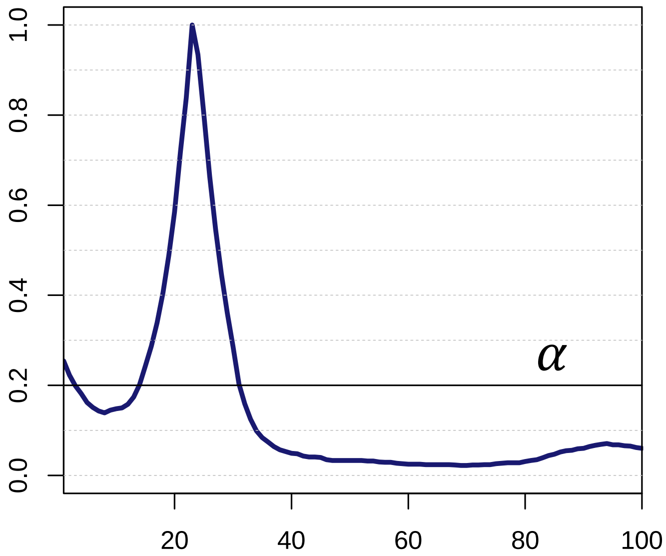
A. Adjust the threshold

$$p(t) < \alpha^*, \quad \text{with} \quad \alpha^* = \arg\max_{\bar{p}} \left\{ \frac{\alpha \mu\{r : p(r) \leq \bar{p}\}}{\mu(D)} \geq \bar{p} \right\}$$

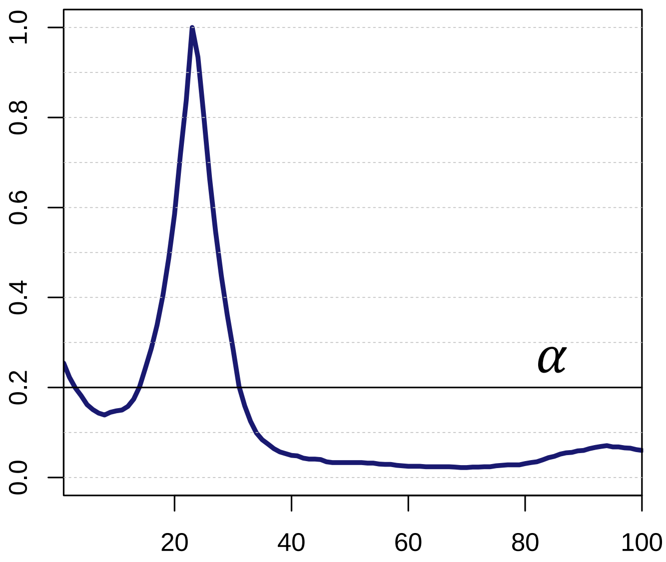
B. Adjust the *p*-value

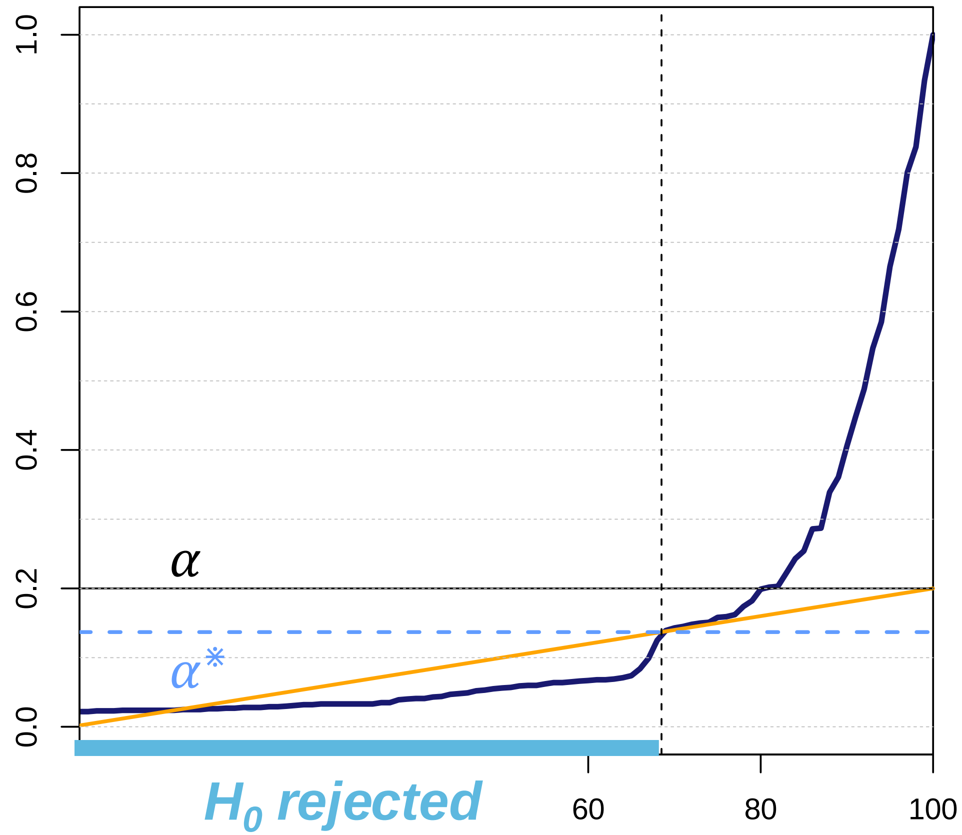
$$\tilde{p}(t) < \alpha, \quad \text{with} \quad \tilde{p}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu\{r : p(r) \leq s\}} \right\}$$

ADJUSTING THE THRESHOLD



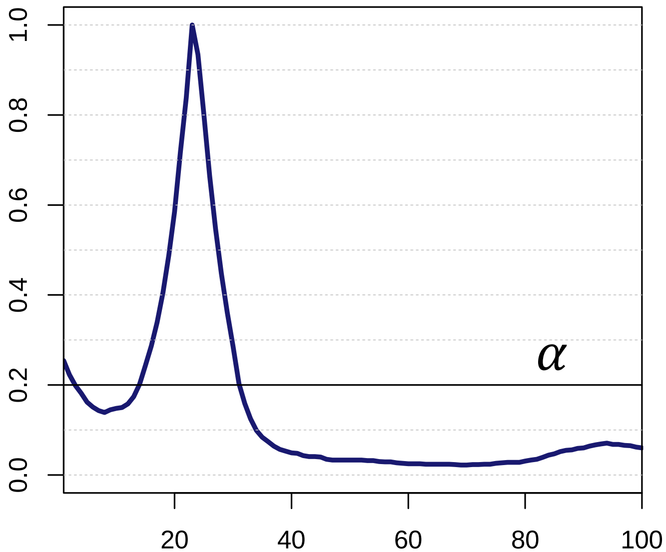
ADJUSTING THE THRESHOLD

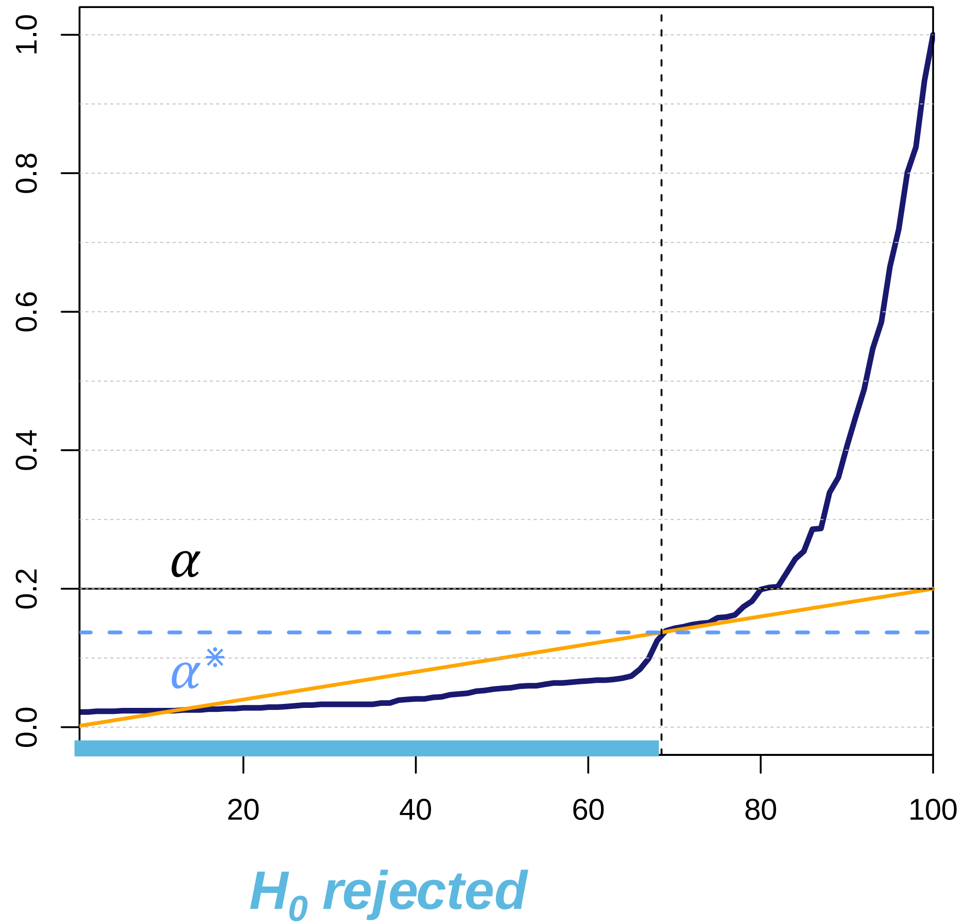


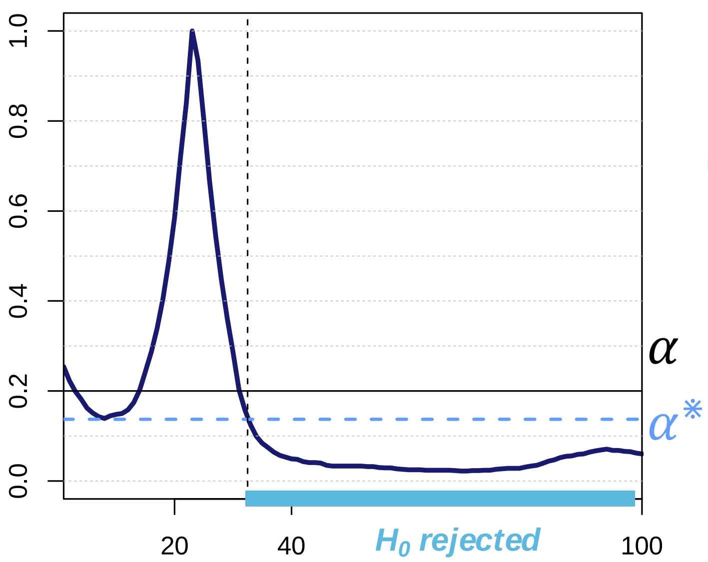


$$\alpha^* = \arg\max_{\bar{p}} \left\{ \frac{\alpha\mu\{r : p(r) \leq \bar{p}\}}{\mu(D)} \geq \bar{p} \right\}$$

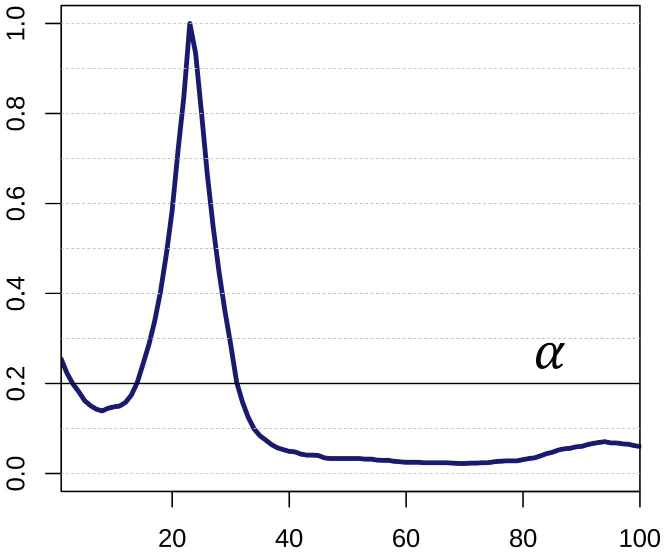
ADJUSTING THE THRESHOLD







ADJUSTING THE P-VALUE FUNCTION



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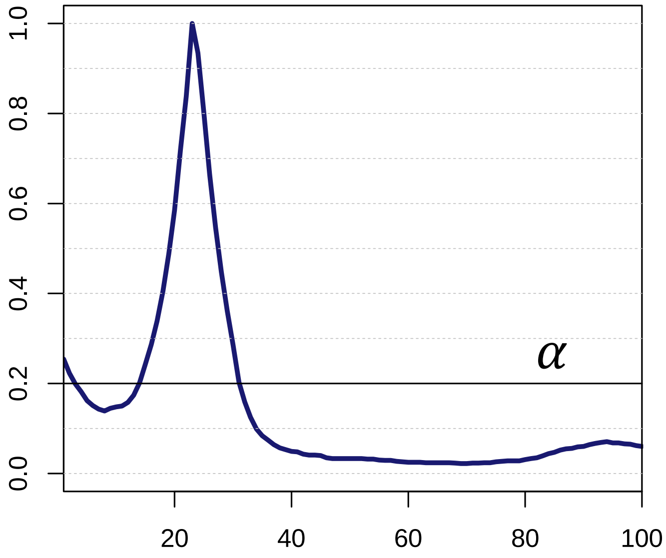

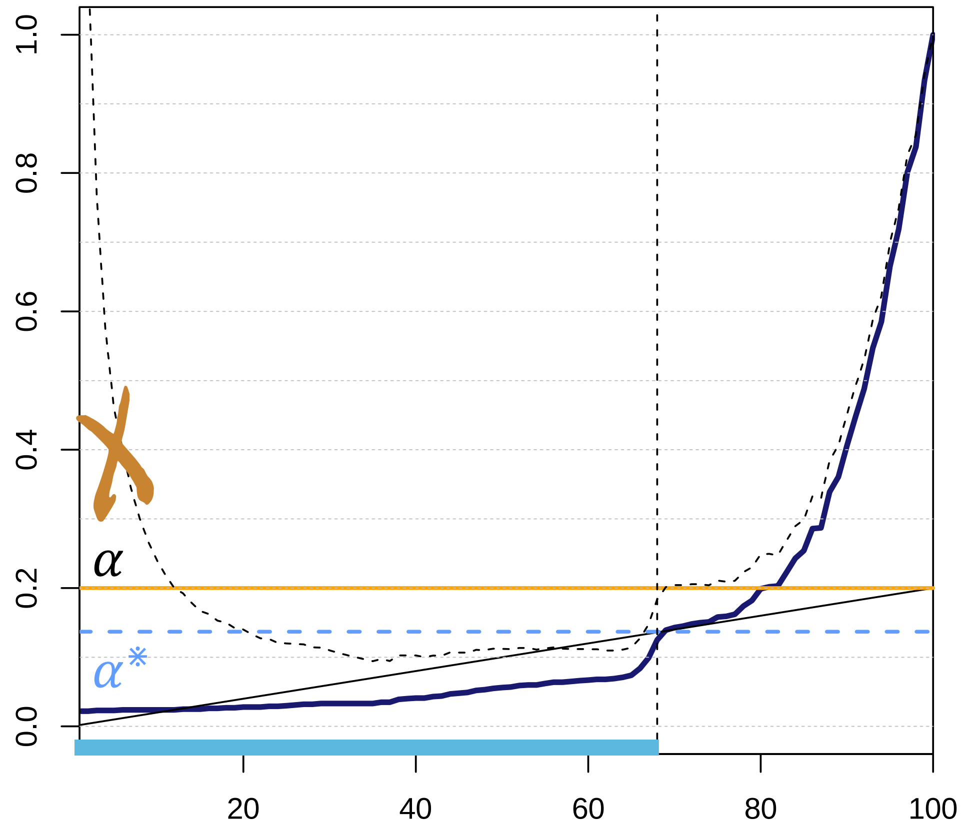

H_0 **rejected**

$$\tilde{p}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu\{r : p(r) \leq s\}} \right\}$$

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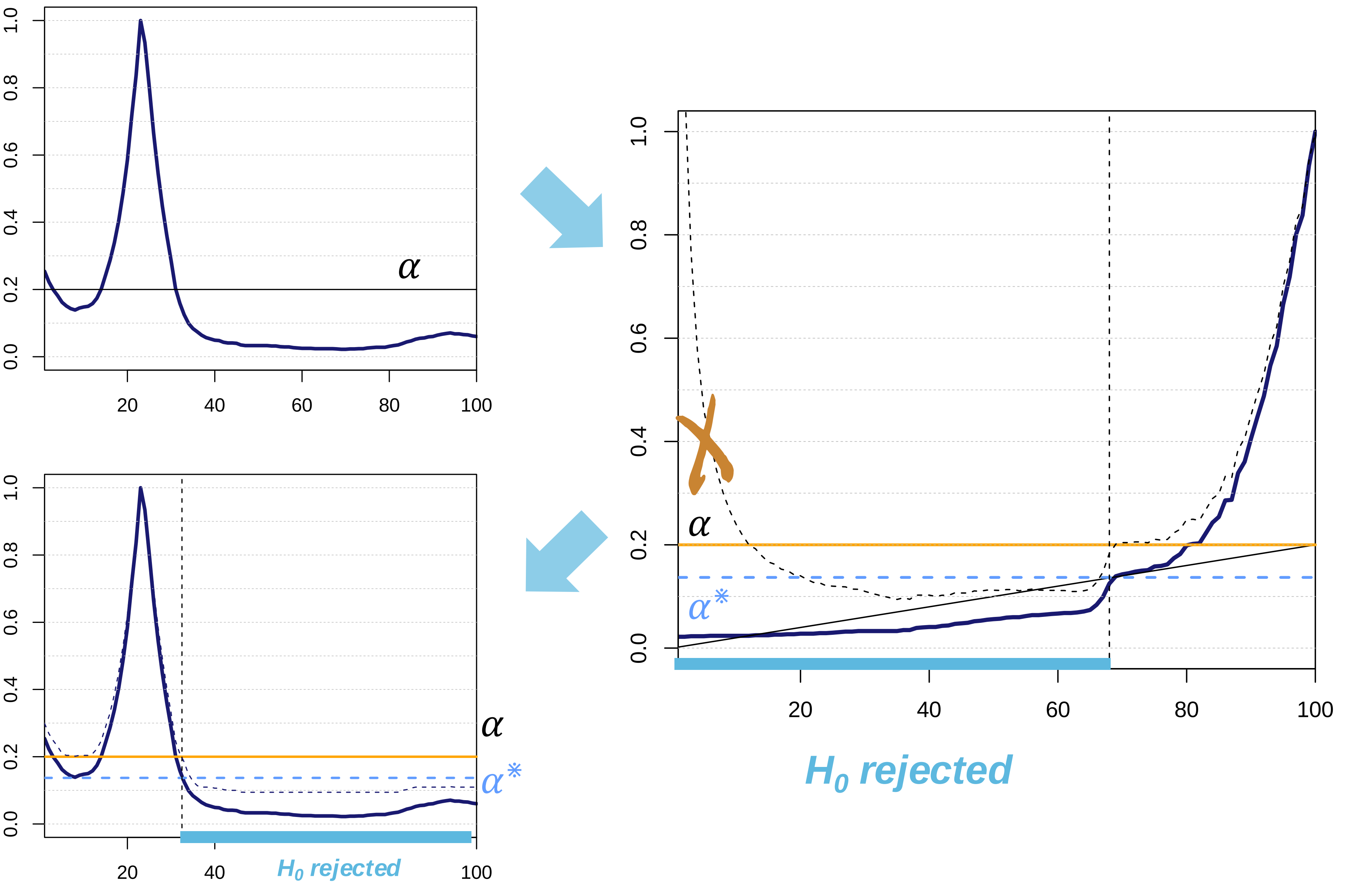
ADJUSTING THE P-VALUE FUNCTION



THE CONTROL OF THE FUNCTIONAL FALSE DISCOVERY RATE

$$\tilde{p}(t) < \alpha, \quad \text{with} \quad \tilde{p}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu\{r : p(r) \leq s\}} \right\}$$

$$p(t) < \alpha^*, \quad \text{with} \quad \alpha^* = \arg\max_{\bar{p}} \left\{ \frac{\alpha\mu\{r : p(r) \leq \bar{p}\}}{\mu(D)} \geq \bar{p} \right\}$$

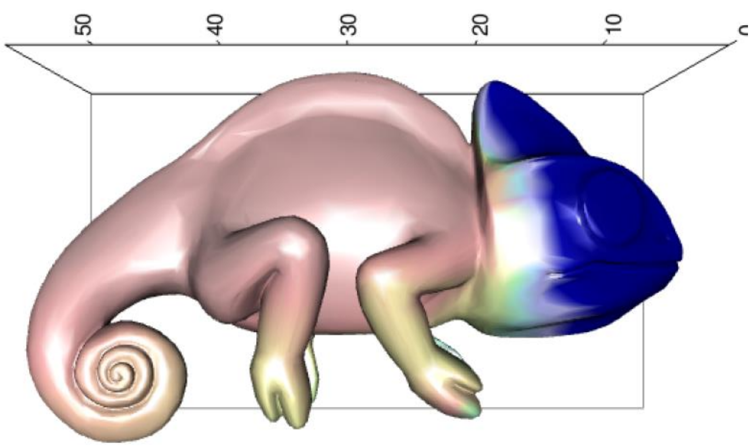
Control of the functional FDR:

$$FDR = \mathbb{E}\left[\frac{V}{R}1(R > 0)\right] \leq \alpha \frac{\mu(D_0)}{\mu(D)} \leq \alpha$$

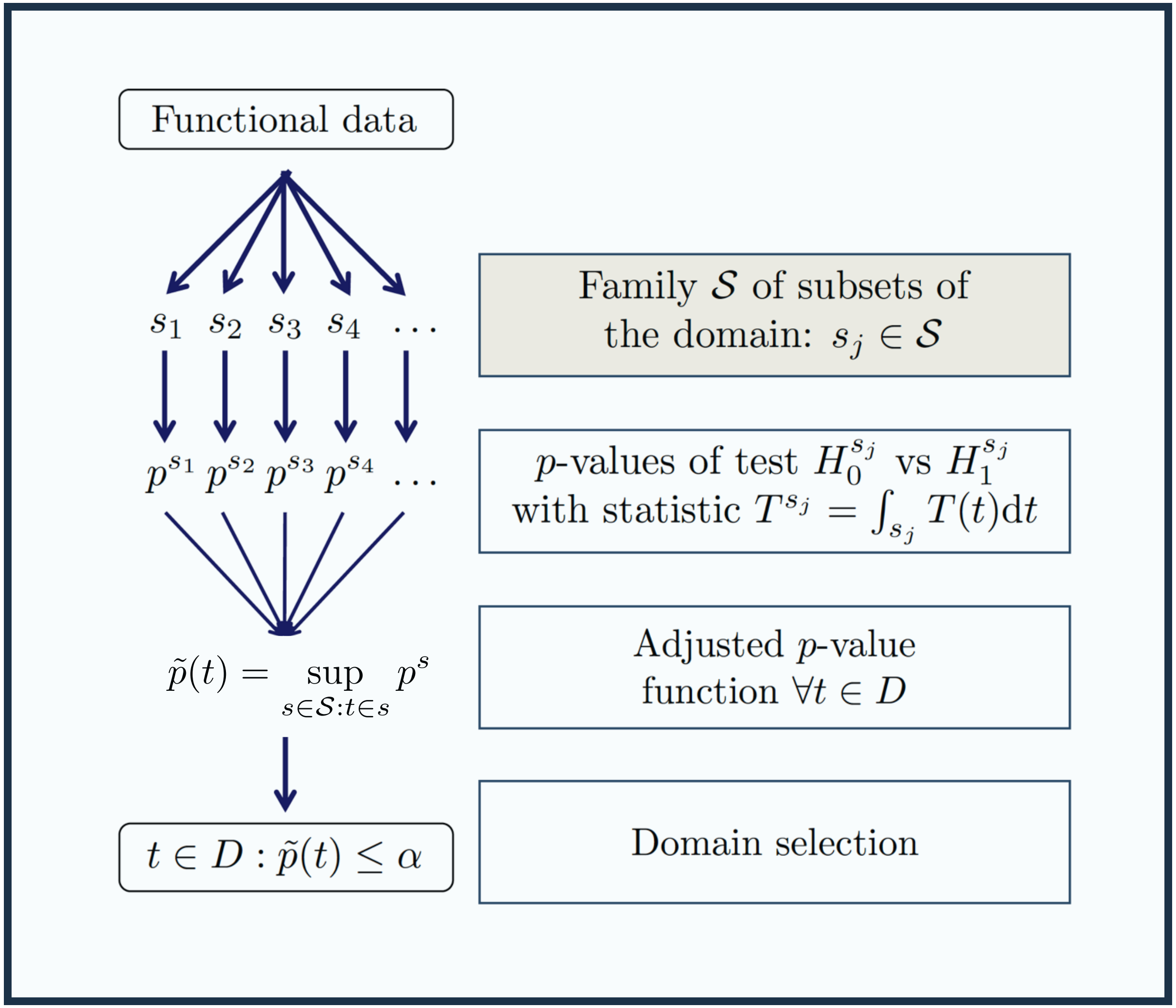
Ball-wise adjusted p-value function

Olsen et al 2025+

$$\tilde{p}(t) = \sup_{\mathcal{I} \ni t} p^{\mathcal{I}}$$



UNIFIED FRAMEWORK FOR LOCAL TESTING



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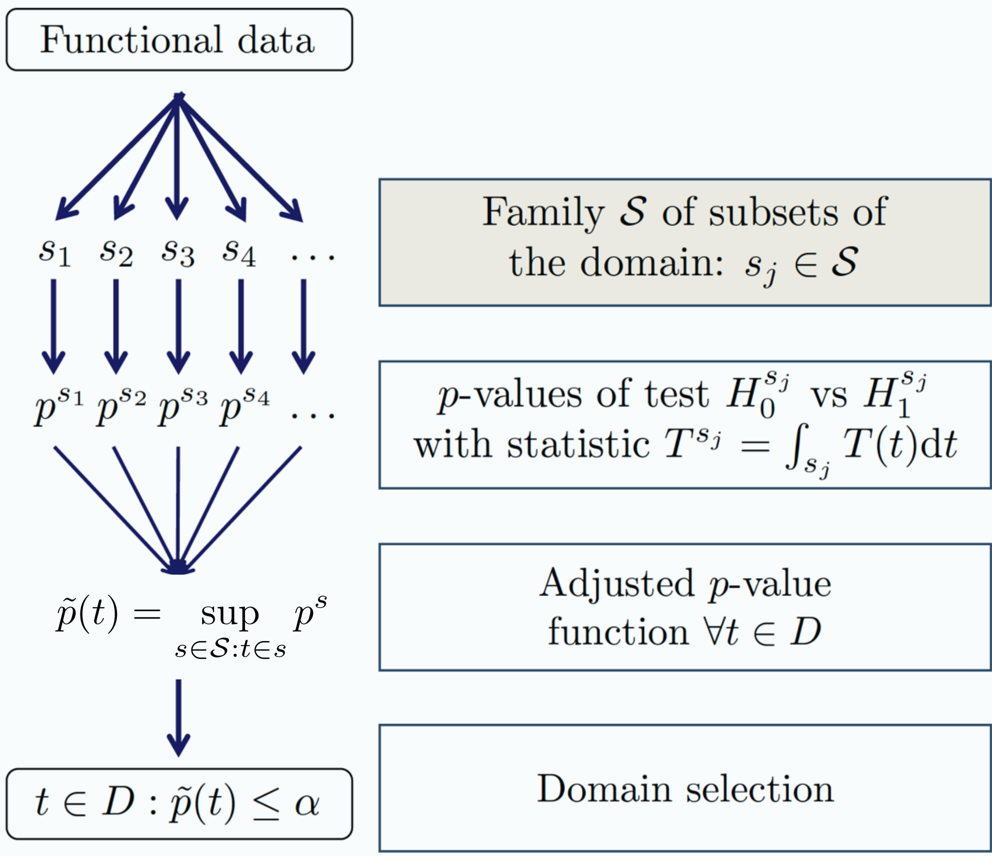


Local adjustment

adjustment subsets should take into account proximity

One dimensional domain: adjustment sets are intervals.

How to choose suitable adjustment subsets when the domain is a manifold?

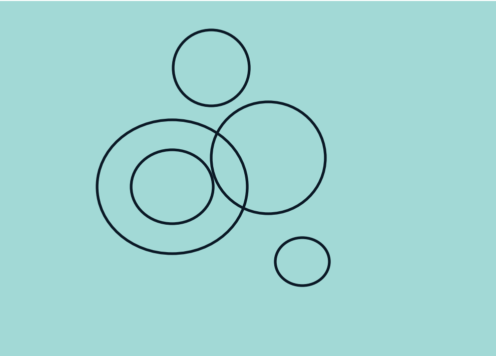
ADJUSTMENT SUBSETS

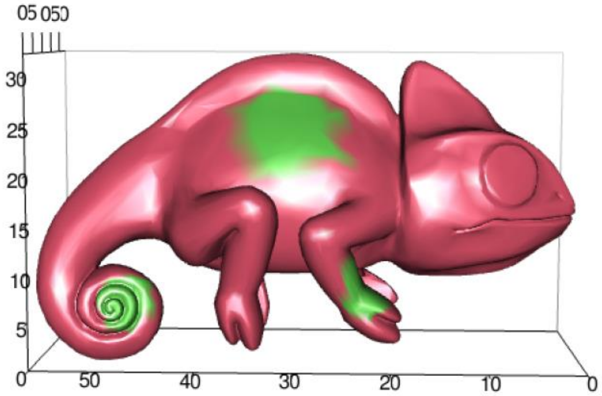
Idea: use balls as adjustment sets.
For a given manifold D with metric d, define $B(x;\epsilon)$ (the ball of center x and radius ϵ) as

$$B(x;\epsilon) = \{y \in D | d(x;y) < \epsilon\}\,;\; x \in D; \epsilon > 0$$

The adjustment family $\mathcal{S}$ consists of the collection of all balls of D of radius smaller than a constant r:

$$\mathcal{S} = \{B(x;\epsilon)\}_{x \in D, \epsilon \leq r}$$





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Theorem 1. *$\tilde{p}(x)$ controls the ball-wise error rate:*

$$\forall \alpha \in (0,1), \quad \forall B(x,\epsilon) \subseteq D : H_0^I \text{ is true}, \mathbb{P}(\exists s \in I : \tilde{p}(s) \leq \alpha) \leq \alpha$$

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ICE COVER DATA ANALYSIS

Example. Yearly measurements (1987-2015) of ice cover on the northern hemisphere, as measured by the satellites of Copernicus Programme. The aim is to test for significant change in ice cover during the period.

$$y_i(x) = a(x) + b(x)t_i + \varepsilon_i(x)$$

$$H_0 : b(x) = 0 \; \forall x \in D; \qquad H_1 : b(x) \neq 0 \text{ for some } x \in D$$

PROBLEM FORMULATION

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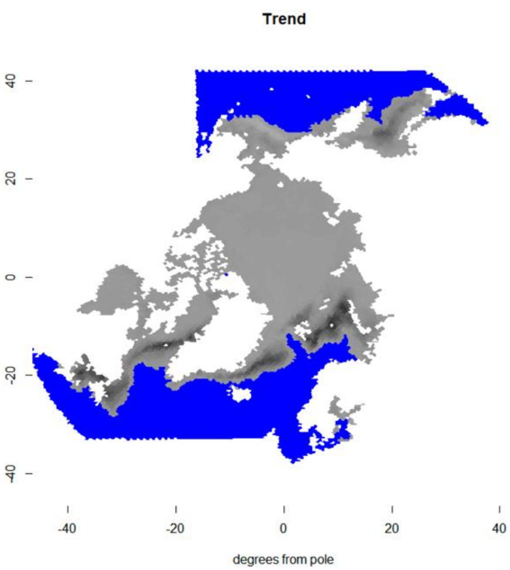
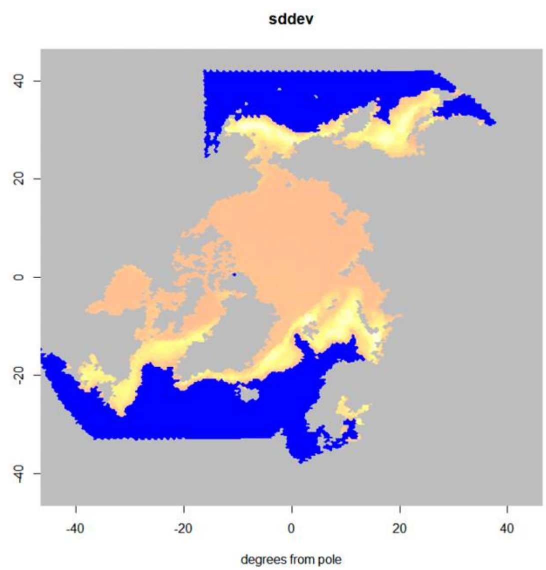
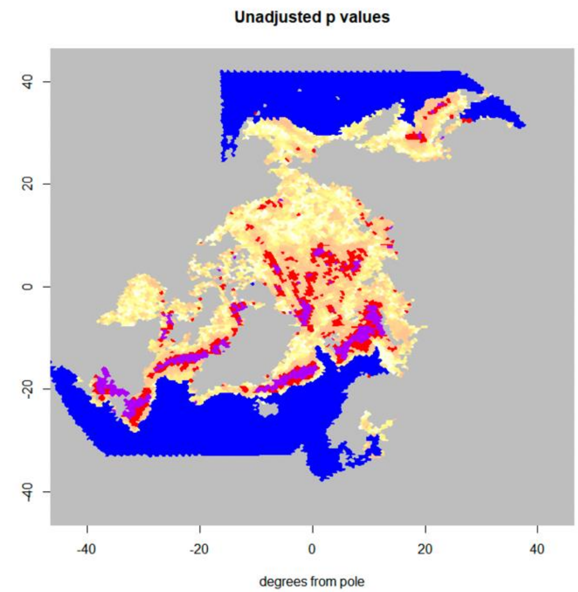




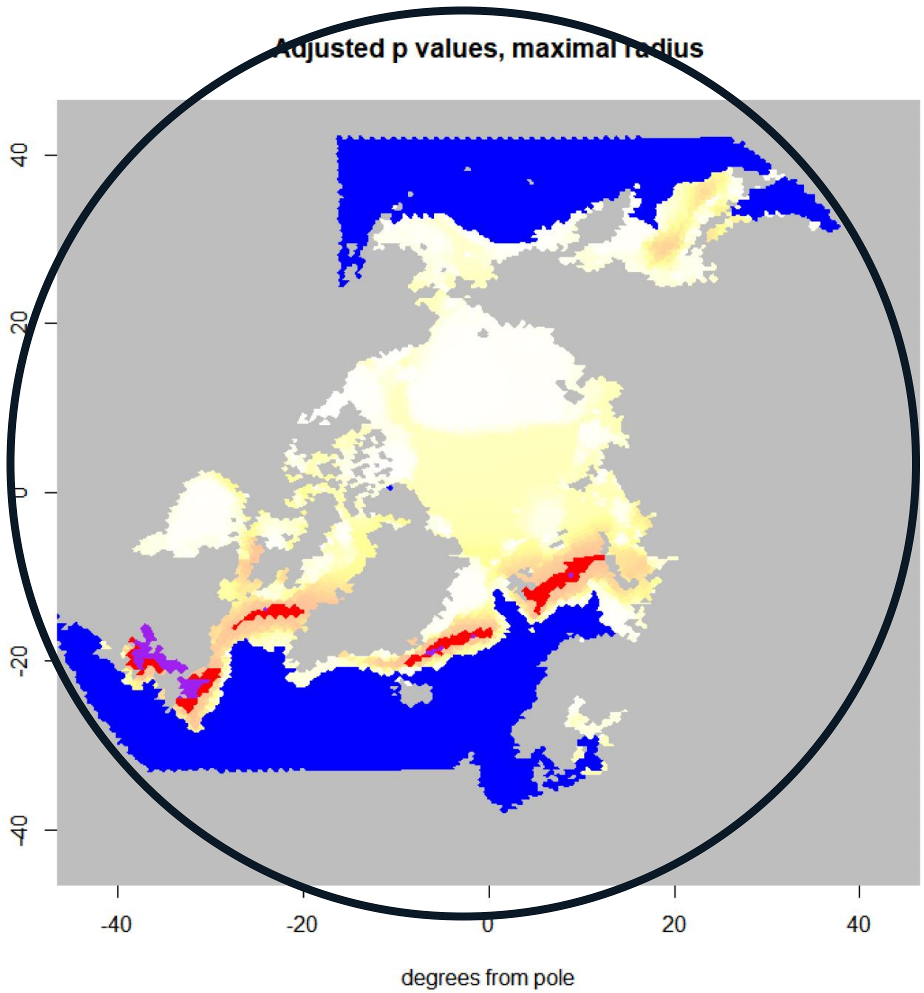


Constant zero ice cover
Negative trend
Zero trend

Constant zero ice cover
Small sd
Large sd

Constant zero ice cover
p-value smaller than 5%
p-value smaller than 1%

ICE COVER DATA ANALYSIS



Constant zero ice cover

Adjusted p-value smaller than 5%

Adjusted p-value smaller than 1%

DISCUSSION

- A method for local inference of functional data with manifold domains.
- A novel way to adjust for multiplicity when the domain is complex.
- The adjustment methods can be plugged-in with every available testing procedure.

Future works

- Are balls the only available type of adjustment sets for local adjustment?
- Is it possible to tune r with a data driven approach?
- Is it possible to develop a local adjustment for the false discovery rate?

Niels Lundtorp Olsen, Alessia Pini, and Simone Vantini. False discovery rate for functional data. *Test*, 2021.

Niels Lundtorp Olsen, Alessia Pini, and Simone Vantini. Local inference for functional data on manifold domains using permutation tests https://arxiv.org/abs/2306.07738

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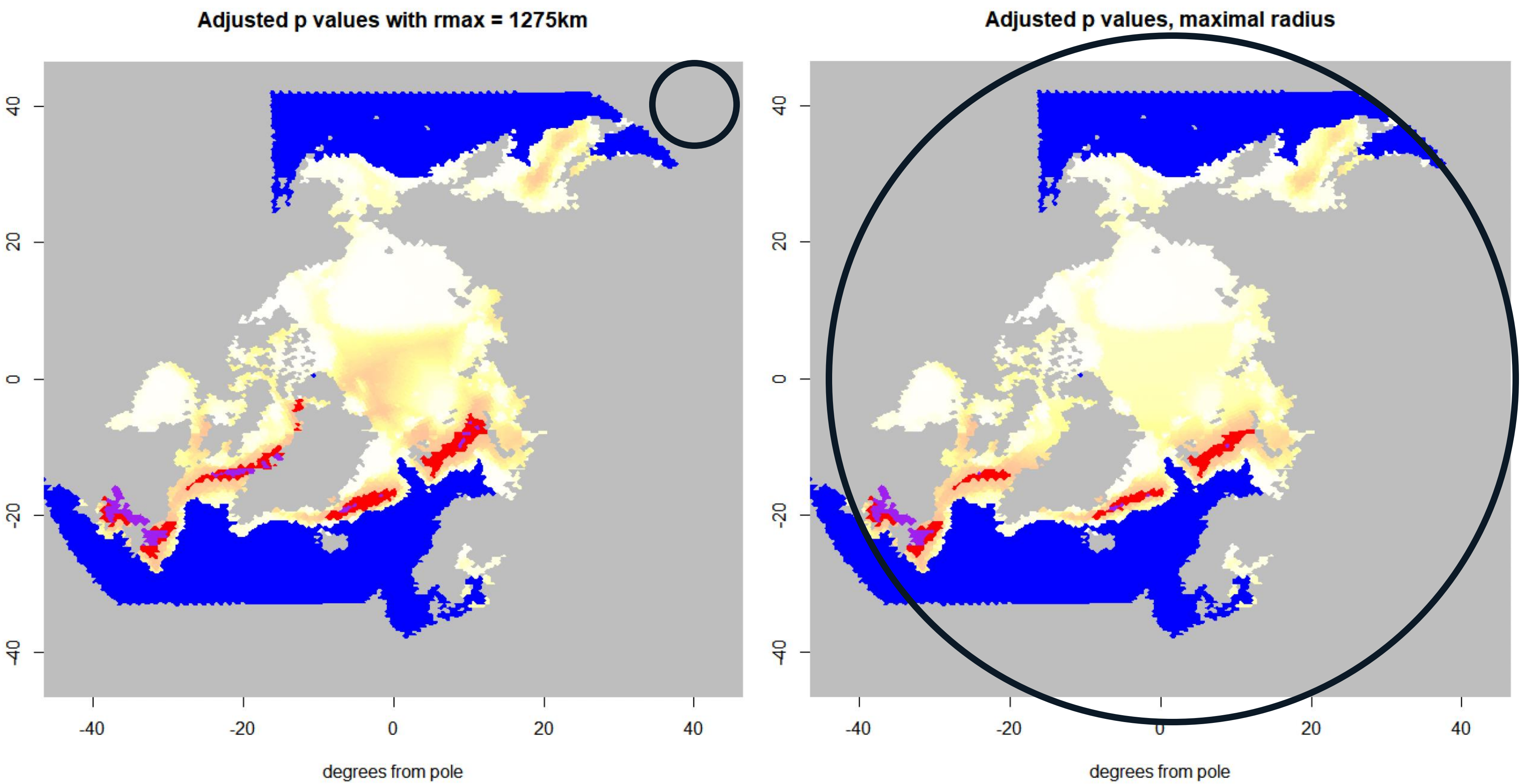
THANK YOU FOR YOUR ATTENTION!

ICE COVER DATA ANALYSIS



Constant zero ice cover
Adjusted p-value smaller than 5%
Adjusted p-value smaller than 1%

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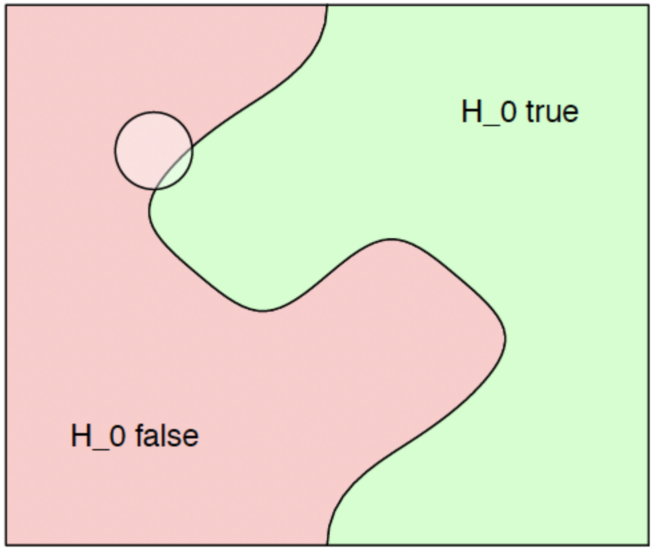
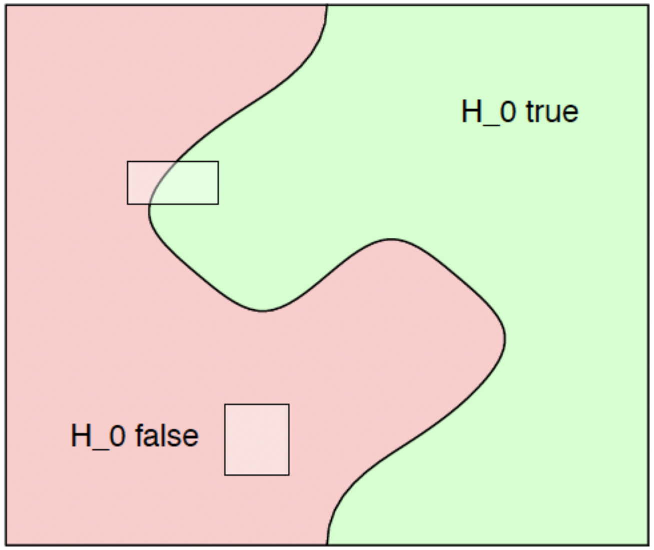
EFFECT OF THE METRIC

- The control that is obtained is strictly related to the type of adjustment sets that are used, which depends on the metric.
- By changing it, one can define different adjustment sets:

Spatial domain



Spatiotemporal domain



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EFFECT OF THE RADIUS

- The radius r also influences the adjustment family and hence the control.
- Two extreme cases:

$$r \to 0$$

Adjustment subsets collapse to points. No adjustment is performed and:

$$\tilde{p}(x) = p(x).$$

The power is maximized, but the error control is minimal.

$$r \to \infty$$

Adjustment subsets include all possible balls. The procedure is very conservative since the adjustment family is very large. The power is minimized, but the error control is maximal.

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CURRENT LITERATURE OF LOCAL INFERENCE

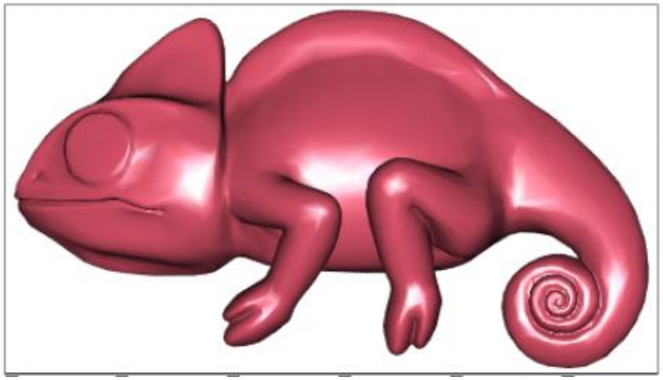
- One dimensional domains
- Manifold domains

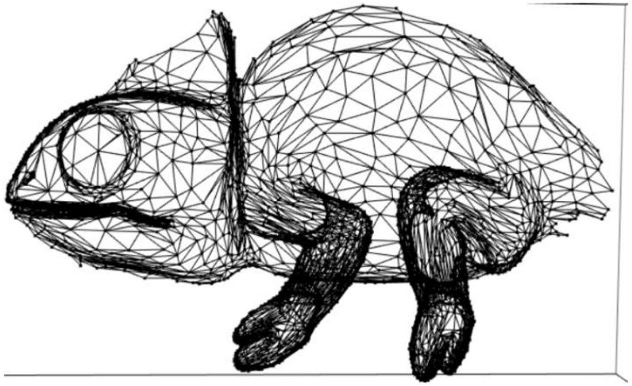
- Global adjustment: the adjustment is based on the pointwise p-values only
- Local adjustment: the adjustment is based on the topological structure of the domain

	One dimensional D	Manifold D
Global adjustment	Multivariate approaches (Holm, Bonferroni, Benjamini Hochberg) Functional FDR (Olsen etal 2021)	**Functional FDR (Olsen etal 2021)** Threshold Wise Testing (Abramowicz etal 2022)
Local adjustment	Interval Wise Testing (Pini Vantini 2017) Functional confidence bands (Liebl Rheimerr 2023) Partition closed testing (Vsevolozhskaya etal 2014)	**Ball Wise Testing**

NUMERICAL IMPLEMENTATION

- Triangulation to approximate the points of the manifold.





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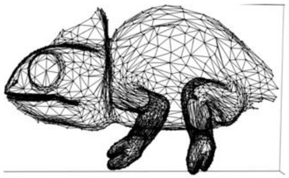
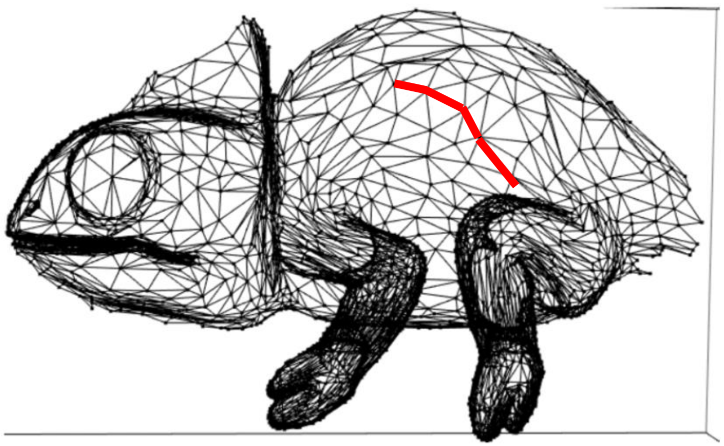


- Dijkstra's algorithm to compute geodesic distance on the manifold.



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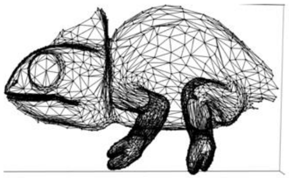

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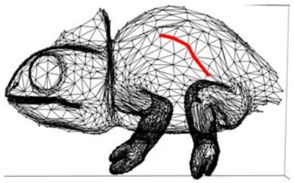

- Integral approximation for computing the test statistic on balls.

$$\int_{B(x,r)} f(u)\,\mathrm{d}u = \sum_{\{v \in E : d(x,e) < r\}} W(e) f(e)$$

$$W(e) = \frac{1}{3} \sum_{S : e \text{ is a vertex of } S} A(S), \quad e \in E$$

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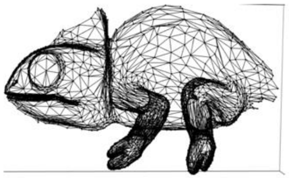

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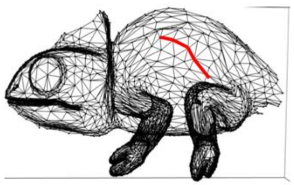

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- Permutation tests to evaluate the p-value of tests on balls.

$$y_i(x) = a(x) + b(x)t_i + \varepsilon_i(x)$$

$$\text{Under } H_0 : y_i(x) = a(x) + \varepsilon_i(x)$$

$$\boxed{\widehat{\varepsilon}_i(x)} = y_i(x) - \widehat{a}(x)$$

Freedman and Lane method: Permutation of the estimated residuals under the null hypothesis