LOCAL NULL HYPOTHESIS SIGNIFICANCE TESTING ON RIEMANIANN MANIFOLDS

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International Workshop
on Functional
and Operatorial Statistics

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MOTIVATION

Functional data on $L^2(D) \cup \mathcal{C}^0(D)$, where $D \subset \mathbb{R}^p$.
We assume that the domain D is a Riemaniann manifold.
Aim: to test a functional hypothesis H_0 against H_1.

Requirements:

- ***Effectiveness***: Solving the Right Problem
- ***Efficiency***: Solving the Problem Optimally
- ***Reliability***: Ensuring Effectiveness Beyond Ideal Conditions
- ***Explainability***: Understanding and Advancing Knowledge

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Requirements:

- ***Effectiveness***: Solving the Right Problem
  - ➡ Test with a correct error control
- ***Efficiency***: Solving the Problem Optimally
  - ➡ Powerful test
- ***Reliability***: Ensuring Effectiveness Beyond Ideal Conditions
  - ➡ Nonparametric permutation test
- ***Explainability***: Understanding and Advancing Knowledge
  - ➡ Local test

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PROBLEM FORMULATION

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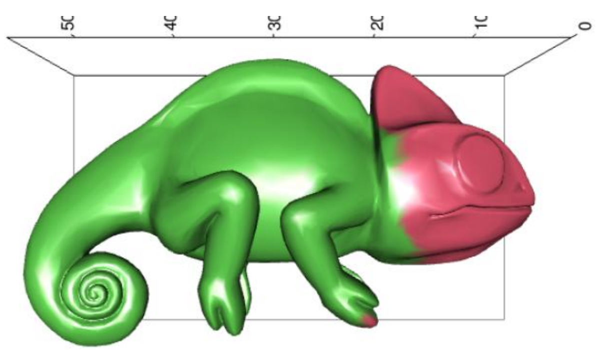
Example. Testing differences between two groups of functional data defined on a complex chamaleon-shaped manifold

$$y_i(x) = \mu_j(x) + \varepsilon_{ij}(x) \quad j = 1, 2;\ i = 1, \ldots, n_j$$

$$H_0 : \mu_1(x) = \mu_2(x)\ \forall x \in D; \qquad H_1 : \mu_1(x) \neq \mu_2(x) \text{ for some } x \in D$$



PROBLEM FORMULATION

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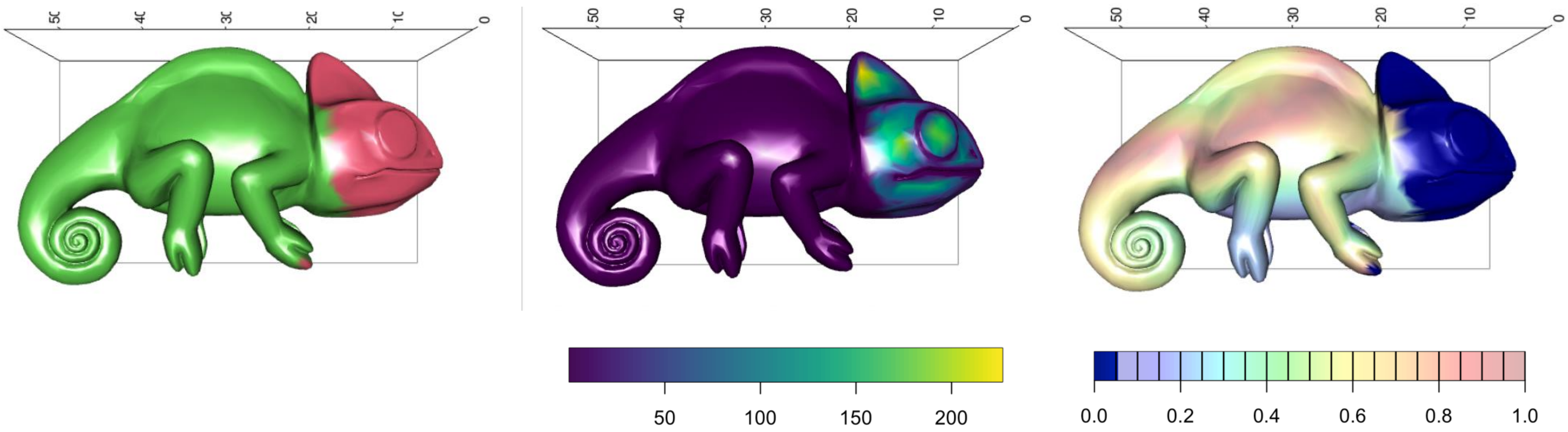
True mean difference | Pointwise t-test statistic | Pointwise p-value



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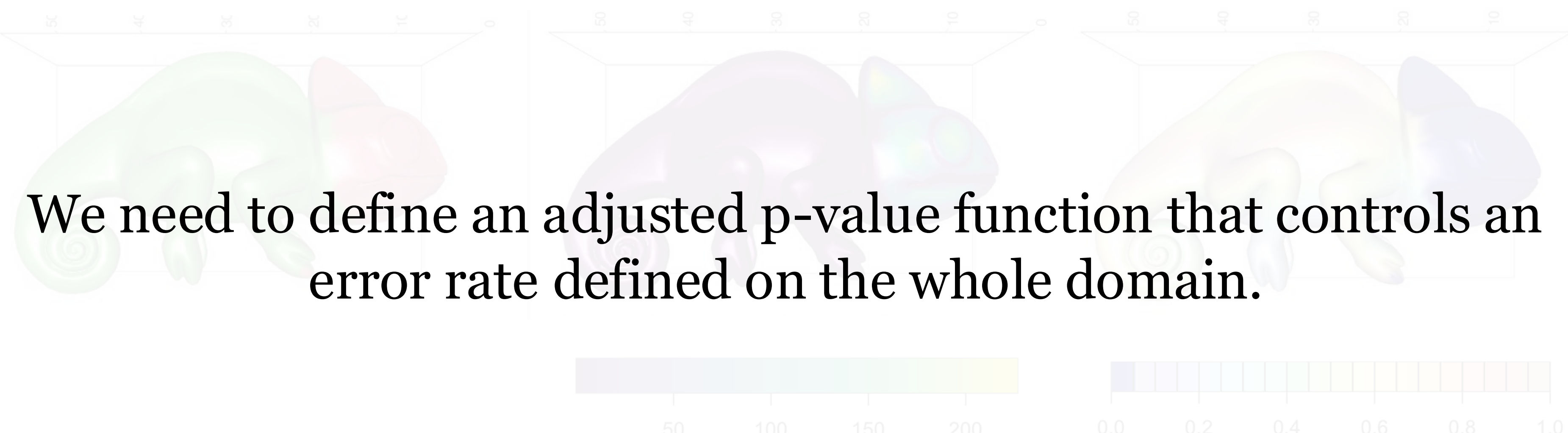
PROBLEM FORMULATION

tional data on $L^2(D) \cap C^0(D)$, where $D \subset \mathbb{R}^p$.

: test **locally** a functional hypothesis H_0 against H_1.

Multiplicity issue!
no control of the amount of type I errors over the domain.

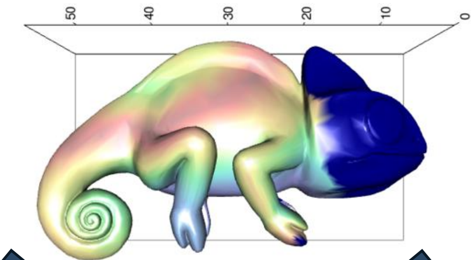
True mean difference | Pointwise t-test statistic | Pointwise p-value



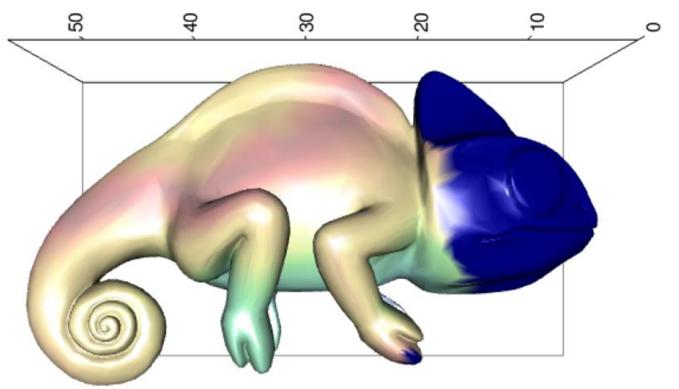
We need to define an adjusted p-value function that controls an error rate defined on the whole domain.

ADJUSTED P-VALUE FUNCTIONS

Pointwise p-value

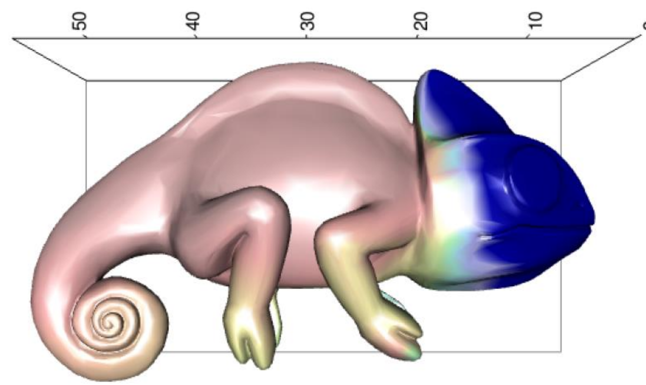


fFDR adjusted p-value function

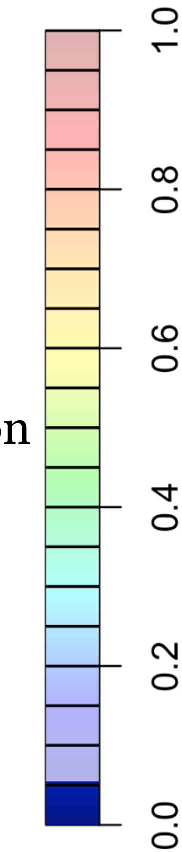


Controls the functional False Discovery Rate

Ball-wise adjusted p-value function



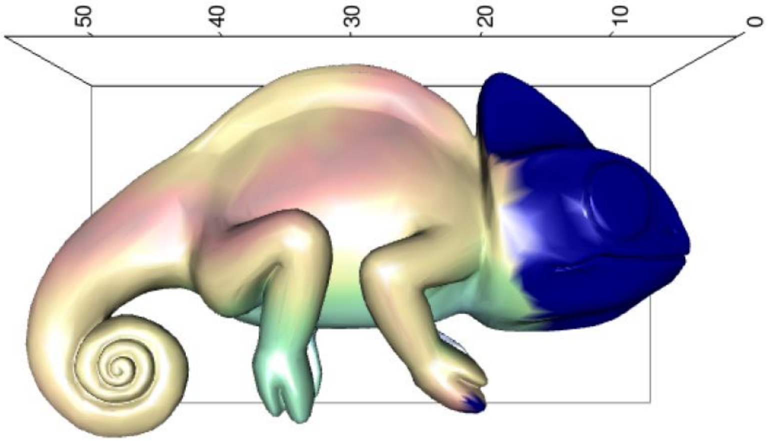
Controls the functional Ball-wise Error Rate



fFDR-adjusted p-value function

Olsen et al 2021 TEST

$$\tilde{p}_{\mathrm{fFDR}}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu(r : p(r) \leq s)} \right\}$$



THE FUNCTIONAL FALSE DISCOVERY RATE

The FDR in multivariate statistics (Benjamini and Hochberg, 1995).

TABLE 1
Number of errors committed when testing m null hypotheses

	Declared non-significant	*Declared significant*	*Total*
True null hypotheses	**U**	**V**	m_0
Non-true null hypotheses	**T**	**S**	$m - m_0$
	$m - \mathbf{R}$	**R**	m

$$\mathrm{FDR} = \mathbb{E}\left[\frac{V}{R}\,1(R > 0)\right]$$

THE FUNCTIONAL FALSE DISCOVERY RATE

The FDR in functional data analysis.

	Declared non significant	Declared significant	
True null Hypotheses	***U*** $\mu\{t \in D : H_{0t} \text{ true}, \tilde{p}(t) > \alpha\}$	***V*** $\mu\{t \in D : H_{0t} \text{ true}, \tilde{p}(t) \leq \alpha\}$	m_0
False null Hypotheses	***T*** $\mu\{t \in D : H_{0t} \text{ false}, \tilde{p}(t) > \alpha\}$	***S*** $\mu\{t \in D : H_{0t} \text{ false}, \tilde{p}(t) \leq \alpha\}$	m_1
	$m - R$	R	m

$$\text{FDR} = \mathbb{E}\left[\frac{V}{R} 1(R > 0)\right]$$

THE FUNCTIONAL BENJAMINI HOCHBERG PROCEDURE

1. Choose a suited functional test (either parametric or non-parametric).

$$H_0^{\mathcal{I}} : \mathcal{Y}_1^{\mathcal{I}} = \mathcal{Y}_2^{\mathcal{I}} \quad H_1^{\mathcal{I}} : \mathcal{Y}_1^{\mathcal{I}} \neq \mathcal{Y}_2^{\mathcal{I}}$$

2. Compute the unadjusted *p*-value function.

$$p(t) = \limsup_{\mathcal{I} \to t} p^{\mathcal{I}}$$

3. Perform functional FDR procedure:

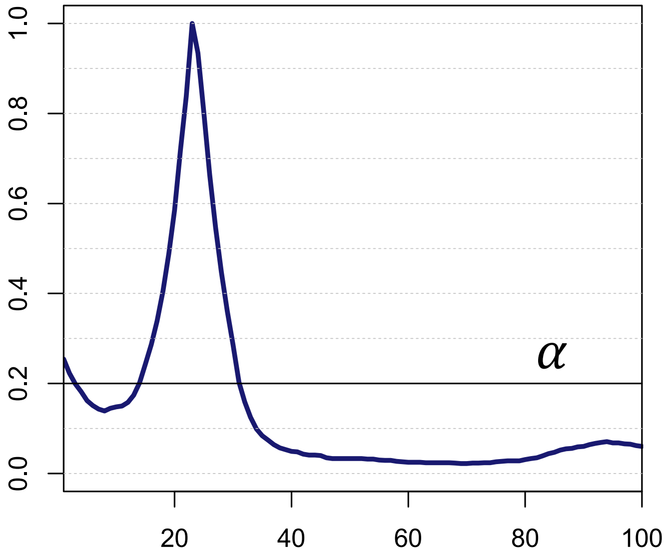
A. Adjust the threshold

$$p(t) < \alpha^*, \quad \text{with} \quad \alpha^* = \arg\max_{\bar{p}} \left\{ \frac{\alpha \mu\{r : p(r) \leq \bar{p}\}}{\mu(D)} \geq \bar{p} \right\}$$

B. Adjust the *p*-value

$$\tilde{p}_{\text{fFDR}}(t) < \alpha, \quad \text{with} \quad \tilde{p}_{\text{fFDR}}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu(r : p(r) \leq s)} \right\}$$

ADJUSTING THE P-VALUE FUNCTION



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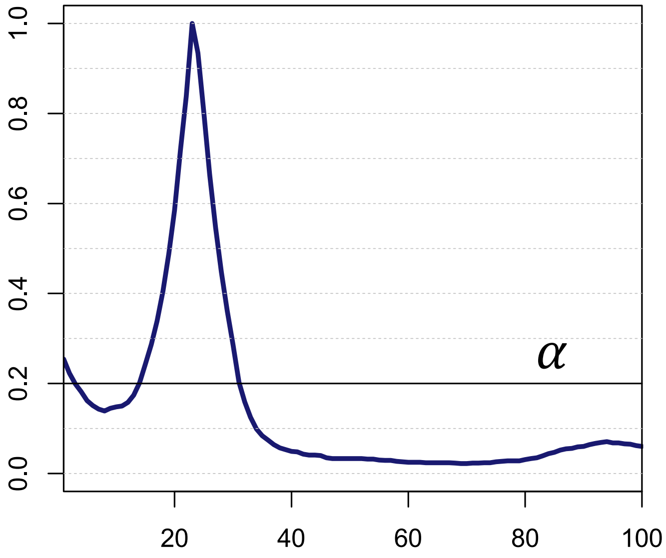
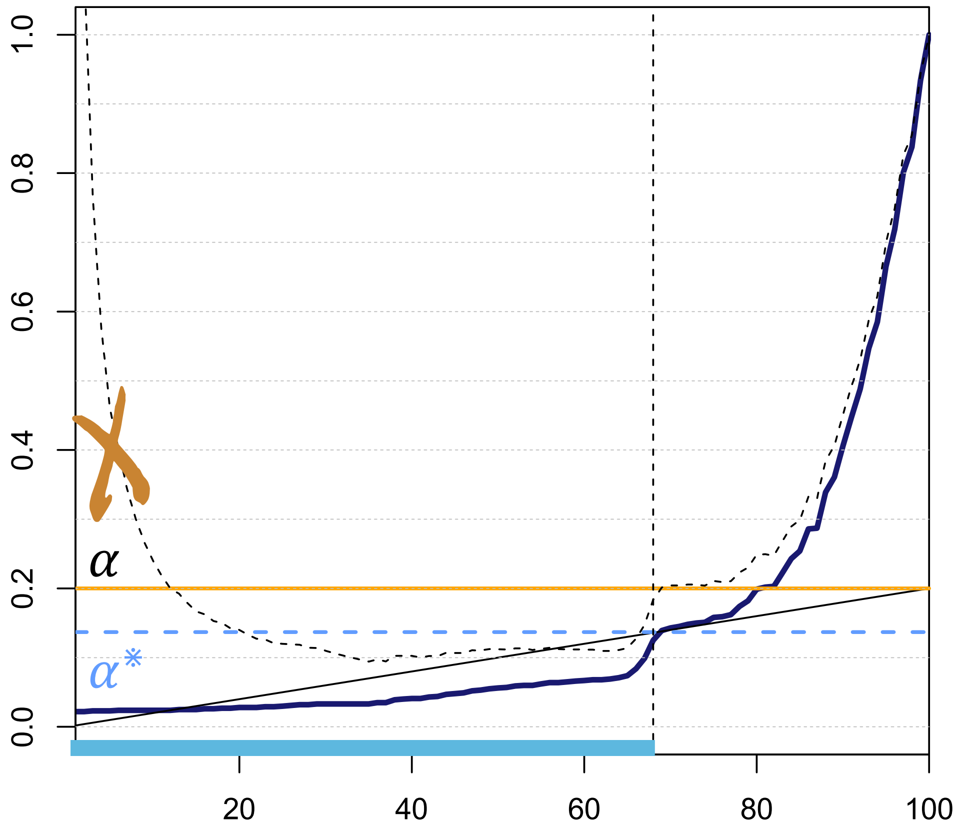
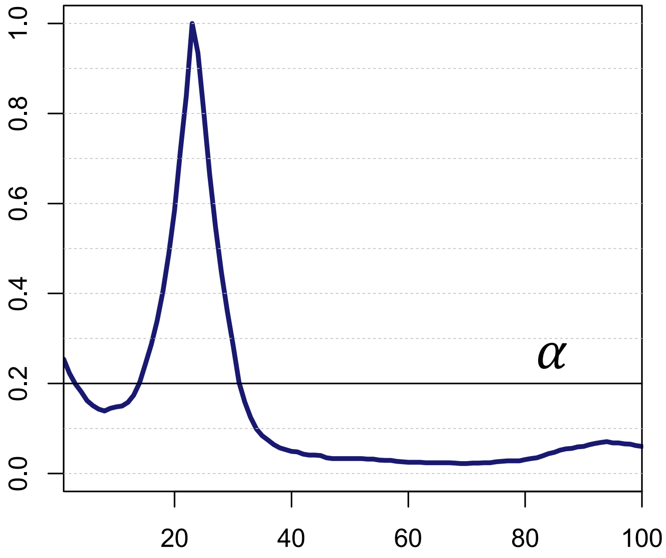


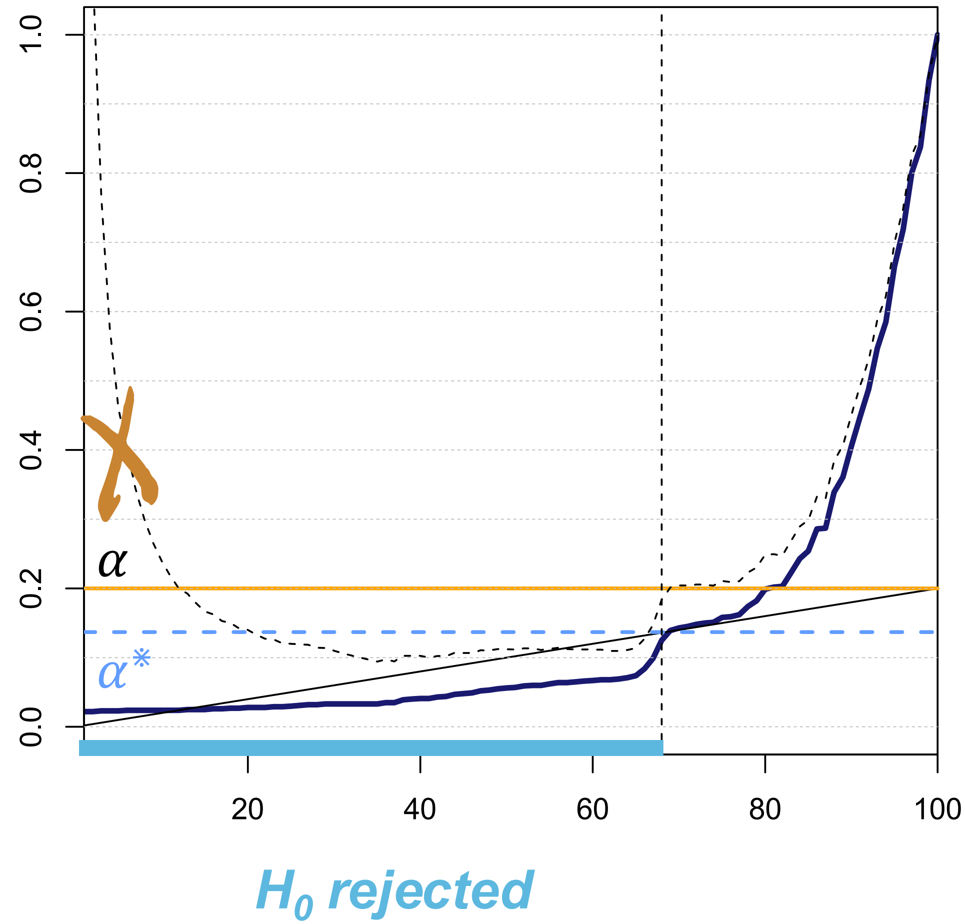


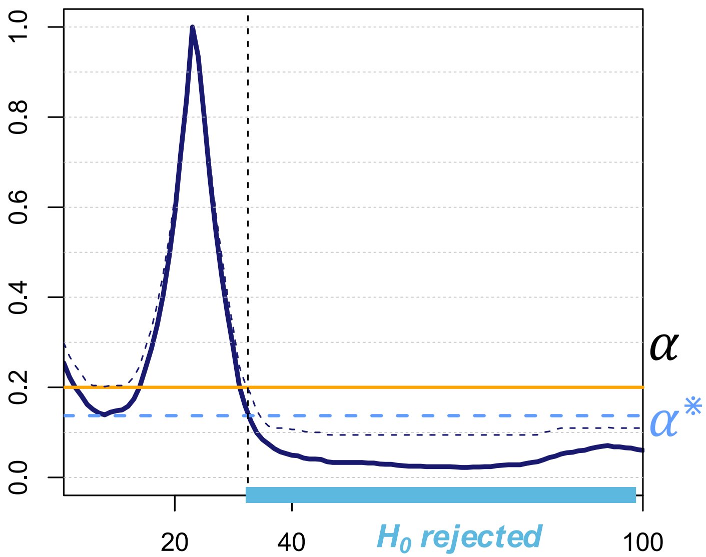
H_0 rejected

$$\tilde{p}_{\mathrm{fFDR}}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu(r : p(r) \leq s)} \right\}$$

ADJUSTING THE P-VALUE FUNCTION







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THE CONTROL OF THE FUNCTIONAL FALSE DISCOVERY RATE

$$\tilde{p}_{\mathrm{fFDR}}(t) = \min_{s \geq p(t)} \left\{ 1, \frac{\mu(D)s}{\mu(r : p(r) \leq s)} \right\}, \quad t \in D$$

The adjusted p-value function $\tilde{p}_{\mathrm{fFDR}}(t)$ controls the fFDR:

$$V_{\tilde{p}} = \{t \in D : H_{0t} \text{ is true } , \tilde{p}_{\mathrm{fFDR}}(t) < \alpha\}$$

$$S_{\tilde{p}} = \{t \in D : H_{0t} \text{ is false } \tilde{p}_{\mathrm{fFDR}}(t) < \alpha\}$$

$$\mathbb{E}\left[\frac{\mu(V_{\tilde{p}})}{\mu(V_{\tilde{p}} \cup S_{\tilde{p}})} 1_{\mu(V_{\tilde{p}} \cup S_{\tilde{p}})>0} \right] \leq \alpha.$$

1921 UN SECOLO DI STORIA DAVANTI A NOI 2021

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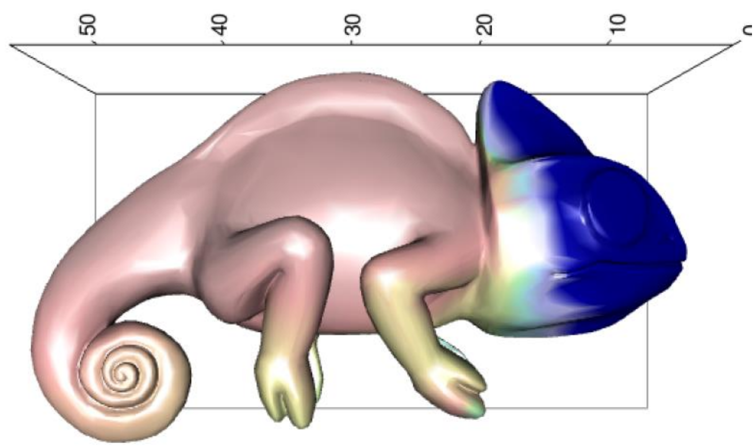
Ball-wise adjusted p-value function

Olsen et al 2025+

$$\tilde{p}_{\text{BWER}}(t) = \sup_{B(x;\epsilon)\in D : t\in B(x;\epsilon)} p^{B(x;\epsilon)}$$



BALL-WISE ADJUSTED P-VALUE FUNCTION

- Ingredients: pointwise hypotheses H_0^t, H_1^t. Pointwise test statistic $T(t)$ defined $\forall t \in D$.

- Start by performing a test of hypothesis on every ball:

$$H_0^{B(x,\varepsilon)} : \bigcap_{t \in \varepsilon} H_0^t \text{ against } H_1^{B(x,\varepsilon)} : \bigcup_{t \in \varepsilon} H_1^t$$

where $B(x;\epsilon) = \{y \in D | d(x;y) < \epsilon\}$; $x \in D; \epsilon > 0$. This can be done using the statistic

$$T^{B(x;\epsilon)} = \int_{B(x;\epsilon)} T(t) \mathrm{d}t.$$

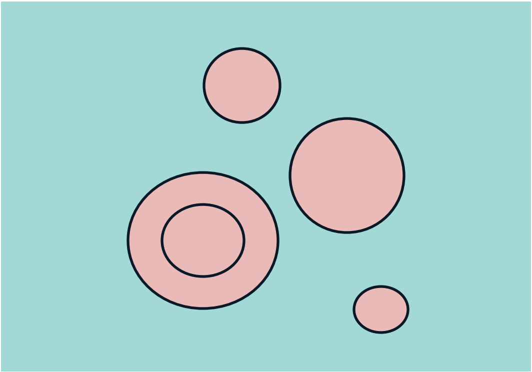
Denote by $p^{B(x,\varepsilon)}$ the corresponding p-value.

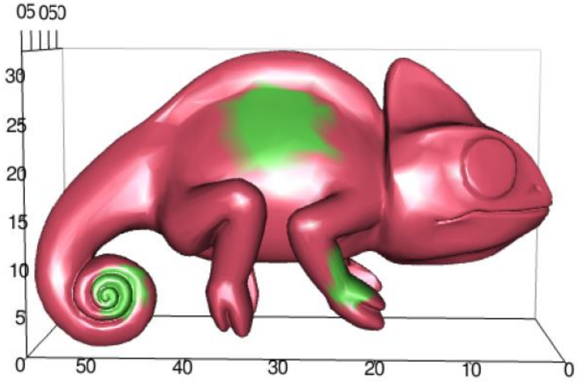
- Compute the adjusted p-value function as

$$\tilde{p}_{\text{BWER}}(t) = \sup_{B(x;\epsilon) \in D : t \in B(x;\epsilon)} p^{B(x;\epsilon)}$$

ADJUSTMENT SUBSETS

Balls are used as adjustment subsets. They are closely related to the metric defined on the manifold:





Balls are also related to the type of error control obtained with this method:

$\tilde{p}_{\mathrm{BWER}}(t)$ controls the ball-wise error rate:

$$\forall \alpha \in (0,1), \quad \forall B(x,\epsilon) \in D : H_0^{B(x,\epsilon)} \text{ is true}$$

$$\mathbb{P}[\exists s \in B(x,\epsilon) : \tilde{p}_{\mathrm{BWER}}(s) \leq \alpha] \leq \alpha$$

ICE COVER DATA ANALYSIS

Example. Yearly measurements (1987-2015) of ice cover on the northern hemisphere, as measured by the satellites of Copernicus Programme. The aim is to test for significant change in ice cover during the period.

$$y_i(x) = a(x) + b(x)t_i + \varepsilon_i(x)$$

$$H_0 : b(x) = 0 \; \forall x \in D; \qquad H_1 : b(x) \neq 0 \text{ for some } x \in D$$

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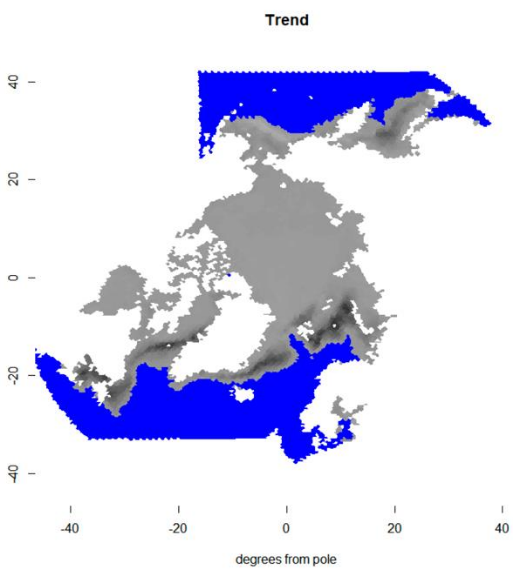
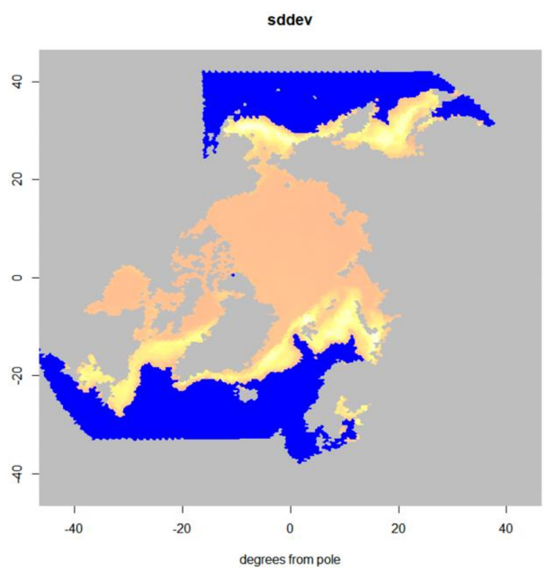
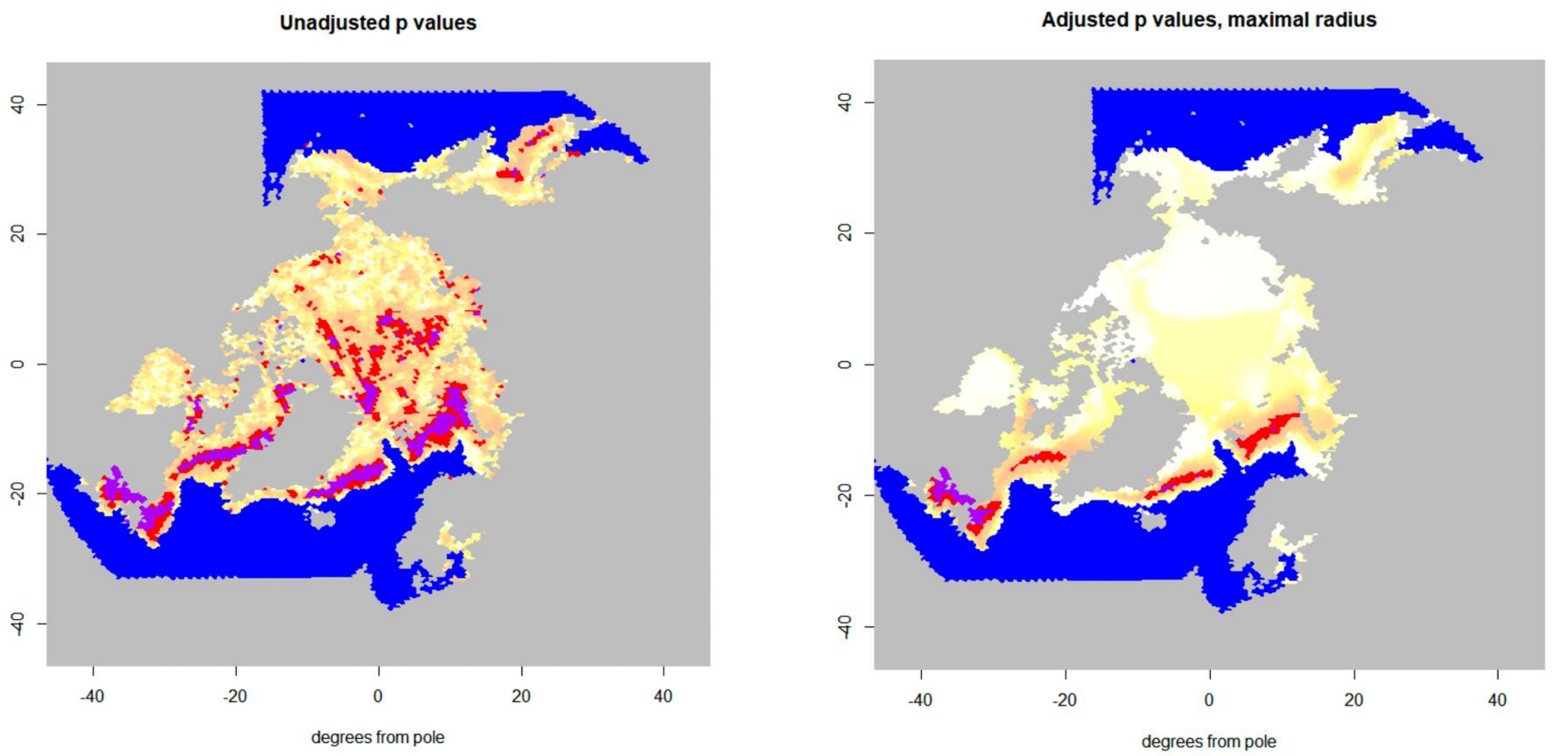




Constant zero ice cover
Negative trend
Zero trend

Constant zero ice cover
Small sd
Large sd

ICE COVER DATA ANALYSIS



Constant zero ice cover
Adjusted p-value smaller than 5%
Adjusted p-value smaller than 1%

DISCUSSION

- Two method for local inference of functional data with manifold domains based on two notions of functional error rate.
- The two adjustment methods can be plugged-in with every available testing procedure.
- Which one to choose in practice?
    - ***Effectiveness***: fFDR or BWER? Problem specific choice
    - ***Efficiency***: fFDR-adjusted p-value is typically more powerful than BWER-adjusted p-value
    - ***Reliability***: both methods are finite sample valid and just rely on exchangeability
    - ***Explainability***: both methods provide an adjusted p-value function for local testing.

Niels Lundtorp Olsen, Alessia Pini, and Simone Vantini. False discovery rate for functional data. *Test*, 2021.

Niels Lundtorp Olsen, Alessia Pini, and Simone Vantini. Local inference for functional data on manifold domains using permutation tests https://arxiv.org/abs/2306.07738

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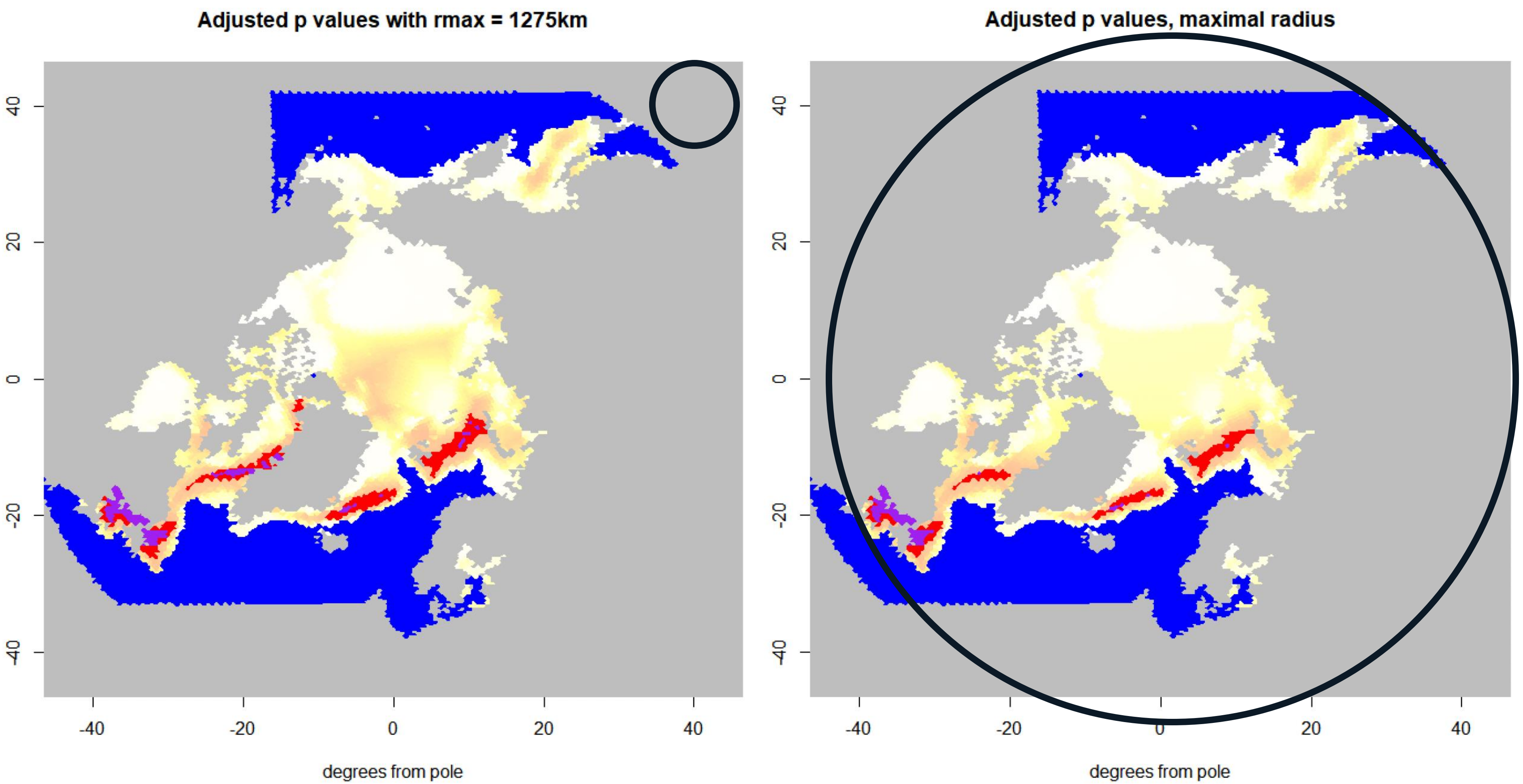
THANK YOU FOR YOUR ATTENTION!

ICE COVER DATA ANALYSIS



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Adjusted p-value smaller than 5%
Adjusted p-value smaller than 1%

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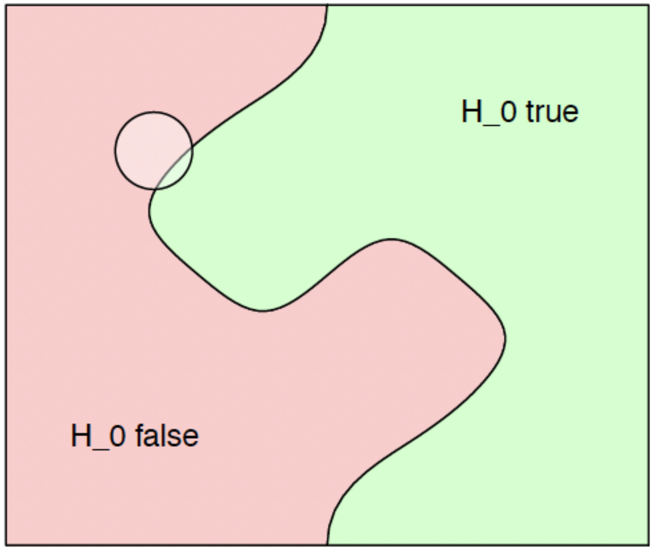
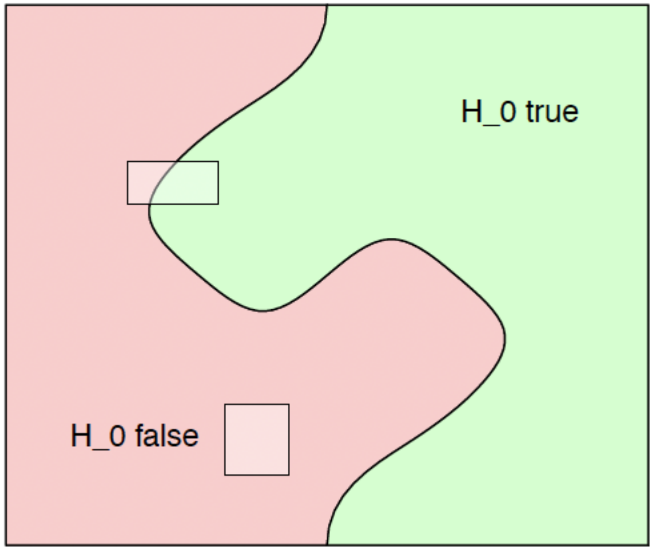
EFFECT OF THE METRIC

- The control that is obtained is strictly related to the type of adjustment sets that are used, which depends on the metric.
- By changing it, one can define different adjustment sets:

Spatial domain



Spatiotemporal domain



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EFFECT OF THE RADIUS

- The radius r also influences the adjustment family and hence the control.
- Two extreme cases:

$r \to 0$

Adjustment subsets collapse to points. No adjustment is performed and:

$$\tilde{p}(x) = p(x).$$

The power is maximized, but the error control is minimal.

$r \to \infty$

Adjustment subsets include all possible balls. The procedure is very conservative since the adjustment family is very large. The power is minimized, but the error control is maximal.

CURRENT LITERATURE OF LOCAL INFERENCE

- One dimensional domains
- Manifold domains

- Global adjustment: the adjustment is based on the pointwise p-values only
- Local adjustment: the adjustment is based on the topological structure of the domain

	One dimensional D	**Manifold D**
Global adjustment	Multivariate approaches (Holm, Bonferroni, Benjamini Hochberg) Functional FDR (Olsen etal 2021)	**Functional FDR (Olsen etal 2021)** Threshold Wise Testing (Abramowicz etal 2022)
Local adjustment	Interval Wise Testing (Pini Vantini 2017) Functional confidence bands (Liebl Rheimerr 2023) Partition closed testing (Vsevolozhskaya etal 2014)	**Ball Wise Testing**

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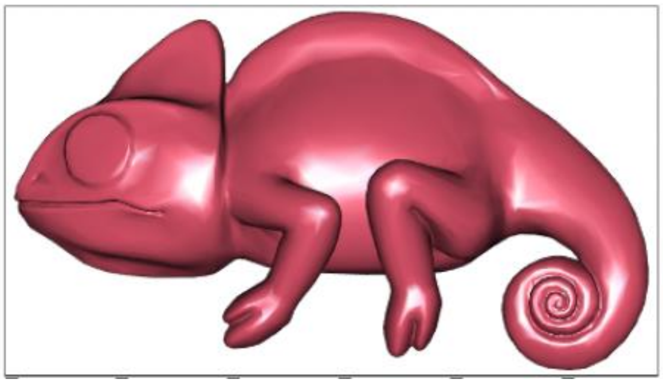
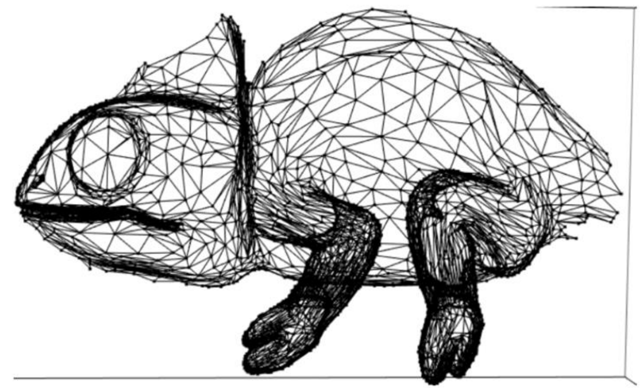
NUMERICAL IMPLEMENTATION

- Triangulation to approximate the points of the manifold.





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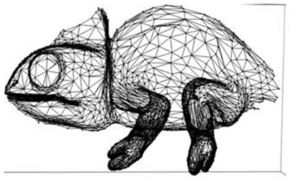
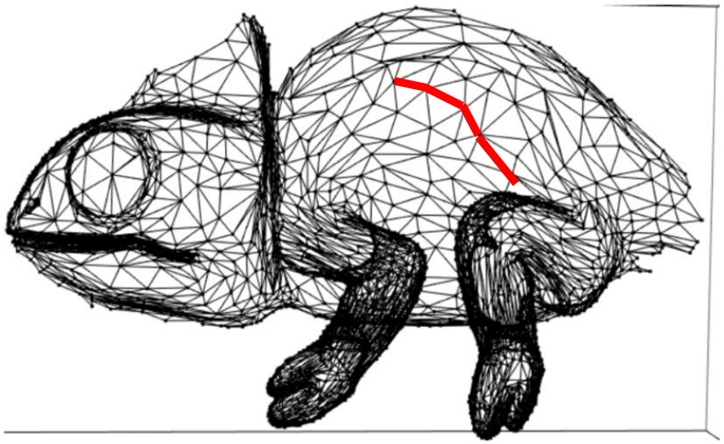


- Dijkstra's algorithm to compute geodesic distance on the manifold.



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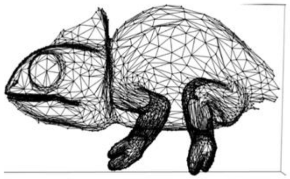

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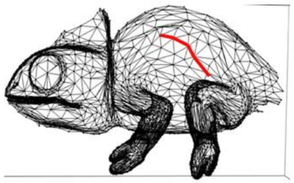

- Integral approximation for computing the test statistic on balls.

$$\int_{B(x,r)} f(u)\,\mathrm{d}u = \sum_{\{v \in E: d(x,e) < r\}} W(e) f(e)$$

$$W(e) = \frac{1}{3} \sum_{S: e \text{ is a vertex of } S} A(S), \quad e \in E$$

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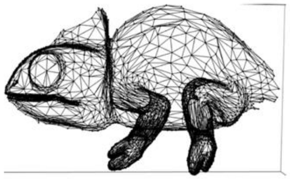

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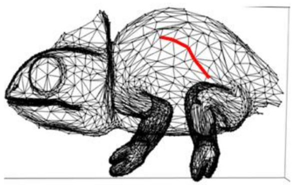

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$$W(e) = \frac{1}{3} \sum_{S: e \text{ is a vertex of } S} A(S), \quad e \in E$$

- Permutation tests to evaluate the p-value of tests on balls.

$$y_i(x) = a(x) + b(x)t_i + \varepsilon_i(x)$$

$$\text{Under } H_0 : y_i(x) = a(x) + \varepsilon_i(x)$$

$$\widehat{\varepsilon}_i(x) = y_i(x) - \widehat{a}(x)$$

Freedman and Lane method: Permutation of the estimated residuals under the null hypothesis