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INFERENCE ON FUNCTIONAL DATA THROUGH E-VALUES

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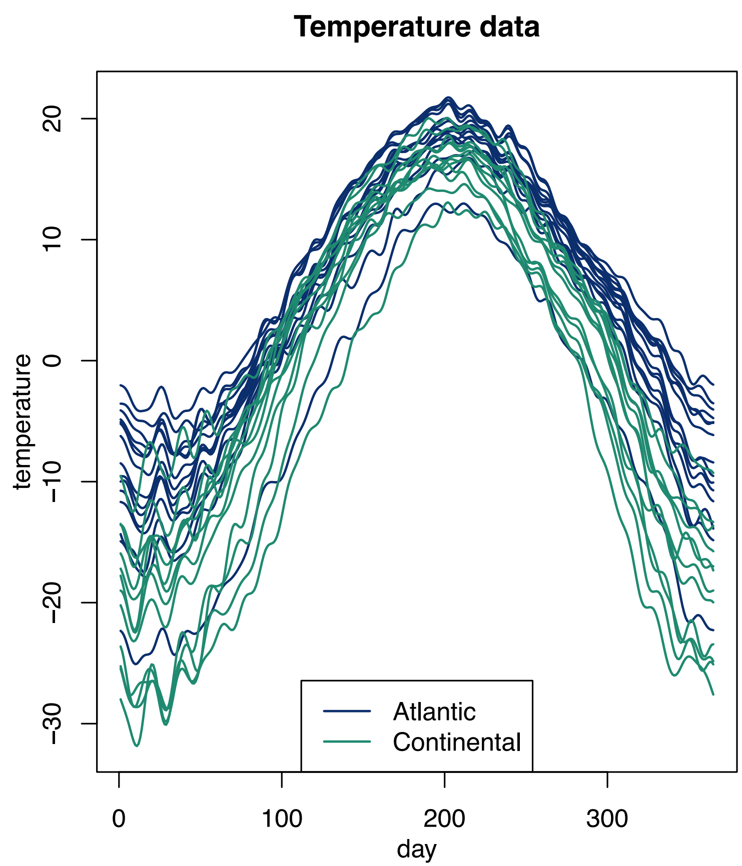
SETTING AND MOTIVATION

We wish to perform statistical hypothesis testing on functional data.

For instance, assume to observe two groups of curves, and wish to test equality in means of the two curves:



Atlantic vs. Continental

SETTING AND MOTIVATION

Formally, assume to observe functional data $\mathcal{X}_i \in L^2(T)$, where $T \subset \mathbb{R}$ with $i = 1, \ldots, n$.
Assume that one wish to perform a null hypothesis significance testing of

$$H_0 \text{ against } H_1.$$

Example: test of difference between two groups of functional data $\mathcal{X}_{ij}$, with $i = 1, \ldots, n_j$, $j = 1, 2$:

$$H_0 : \mathcal{X}_{i1} \stackrel{d}{=} \mathcal{X}_{i2} \text{ against } H_1 : \mathcal{X}_{i1} \stackrel{d}{\neq} \mathcal{X}_{i2}$$

SETTING AND MOTIVATION

It is typically easy to obtain a valid pointwise test, e.g.:

$$H_0^t : \mathcal{X}_{i1}(t) \overset{d}{=} \mathcal{X}_{i2}(t) \text{ against } H_1^t : \mathcal{X}_{i1}(t) \overset{d}{\neq} \mathcal{X}_{i2}(t)$$

However, pointwise tests can not be easily combined to obtain a global *p*-value that is valid on the whole domain!

Possible solutions:

- **Parametric methods**: under assumptions on data distribution and on the covariance structure, one can elicitate the distribution of a gobal test statistic (integral, maximum). ⟵ Strong assumptions
- **Nonparametric methods**: one can perform global tests based on permutations. ⟵ Computationally intensive

SETTING AND MOTIVATION

AIM: to obtain valid global inference by combining pointwise results in a direct way.

✓ No strong assumptions
✓ Computationally efficient

Possible solutions:

- **Parametric methods**: under assumptions on data distribution and on the covariance structure, one can elicitate the distribution of a gobal test statistic (integral, maximum). ⟵ Strong assumptions
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E-VALUES

For a test of hypotheses H_0 against H_1, an e-variable is defined as an extended random variable $E : \Omega \to [0, +\infty]$ such that

$$\mathbb{E}[E] \leq 1 \text{ under } H_0.$$

An e-value is the sample realization of E.

Interpretation:

- Testing hypotheses with e-values can be interpreted as testing by betting: assume that you bet 1 against H_o. The e-value is the betting score: the payoff of this bet.
- A large e-value is evidence against the null hypothesis.
- The larger is e, the stronger is the evidence against H_o.

PROPERTIES OF E-VALUES

E-values can be used to perform tests of hypotheses in the classical sense:

Testing with *e*-values:
Reject H_o if e is large

$$e \geq \frac{1}{\alpha}$$

Testing with *p*-values:
Reject H_o if p is small

$$p \leq \alpha$$

Exactness:
The test that rejects the null hypothesis if $e \geq 1/\alpha$ is exact in the classical sense. In fact, thanks to Markov inequality:

$$\mathbb{P}_{H_0}\left(\frac{1}{E} \leq \alpha\right) = \mathbb{P}_{H_0}\left(E \geq \frac{1}{\alpha}\right) \leq \alpha \qquad \forall \alpha \in (0,1)$$

PROPERTIES OF E-VALUES

Consistency:

Assume that:

$$\lim_{n\to\infty} \mathbb{E}[E] = \infty, \qquad \lim_{n\to\infty} \text{Var}[E] = 0.$$

Then, the test that rejects the null hypothesis if $e \geq 1/\alpha$ is consistent in the classical sense. In fact, again thanks to Markov inequality:

$$\mathbb{P}\left(\frac{1}{E} \leq \alpha\right) = 1 - P\left(\frac{1}{E} > \alpha\right) \geq 1 - \frac{\mathbb{E}(1/E)}{\alpha}$$

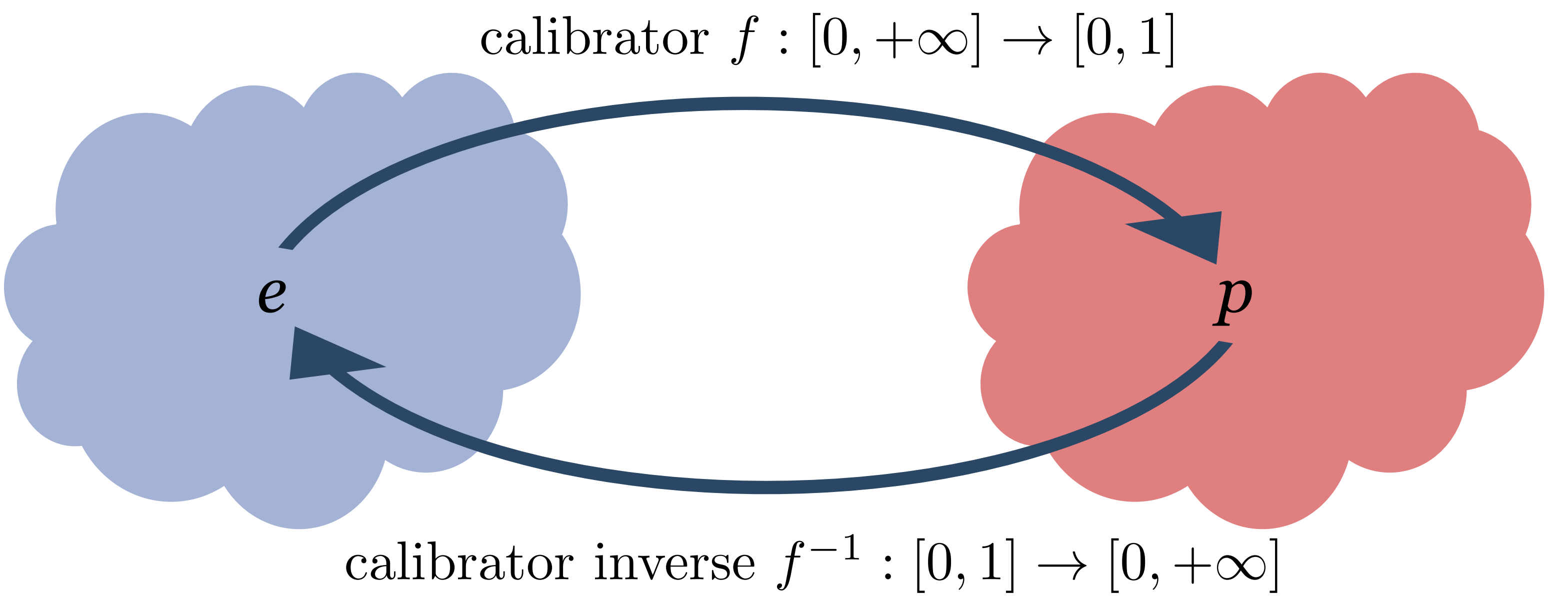
Now, the two assumptions on E imply $\lim_{n\to\infty} \mathbb{E}(1/E) = 0$. Hence:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{E} \leq \alpha\right) = \lim_{n\to\infty} 1 - \frac{\mathbb{E}(1/E)}{\alpha} = 1.$$

RELATIONSHIP BETWEEN E-VALUES AND P-VALUES

If an exact p-value is available for H0 against H1, it is possible to find a corresponding e-value and vice-versa.

There exist several bijective maps from p-values to e-values called calibrators.



WHY E-VALUES?

- E-values can be viewed as an alternative way to test hypotheses.
- It is possible to convert e-values in p-values and vice versa.
- To compute valid e-values is typically easier, since one need to evaluate just the expected value of a test statistic instead of the whole distribution of it.
- E-values can be straightforwardly combined to **test multivariate hypotheses:**

If e_1 and e_2 are e-values for H_{01} and H_{02}, $e_1 + e_2$ is an e-value for $H_{01} \cap H_{02}$.

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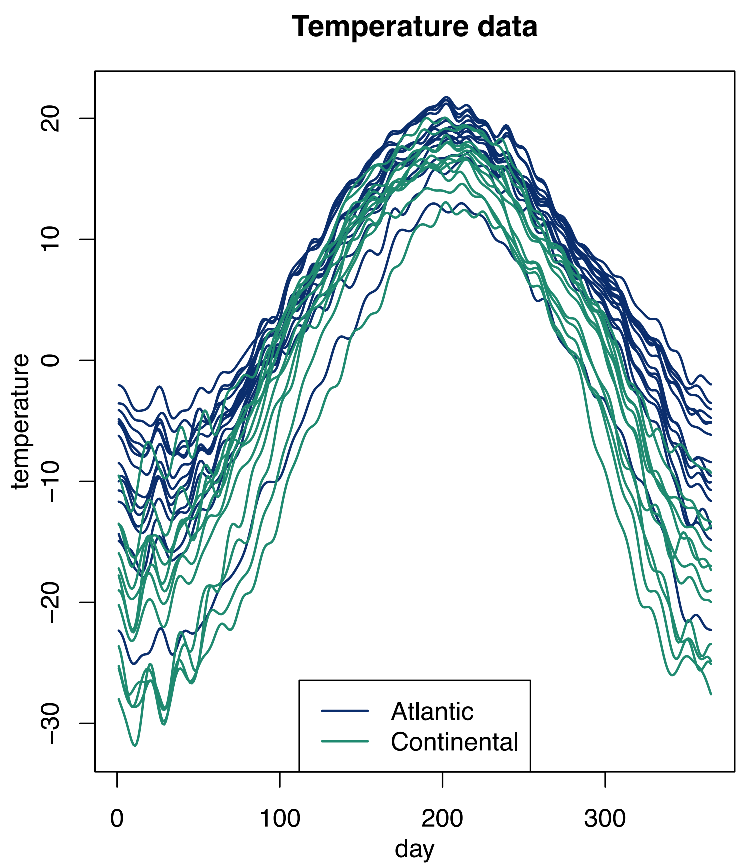
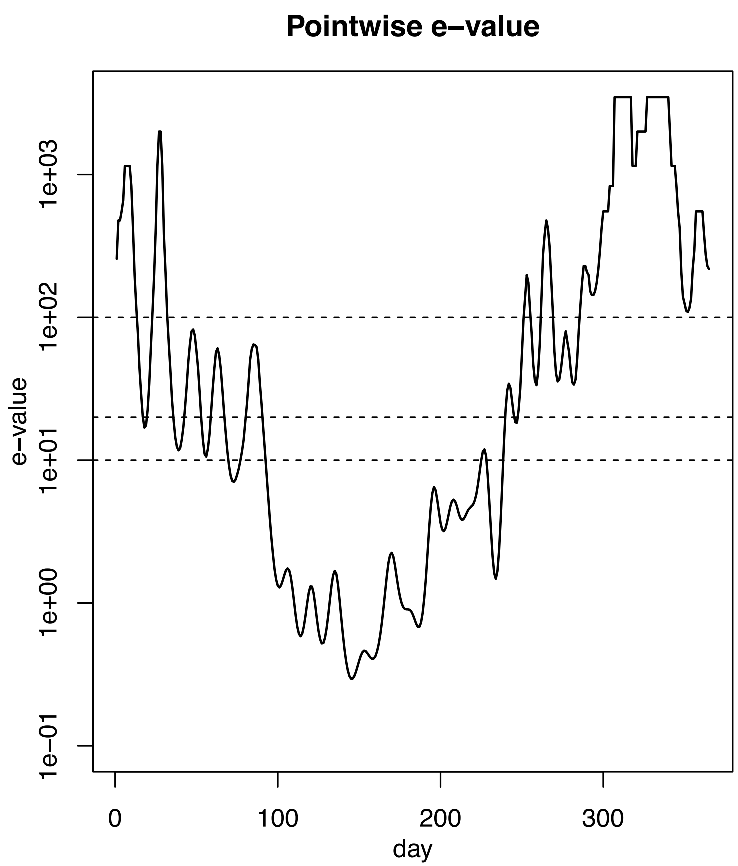
- E-values can be straightforwardly combined to **test multivariate hypotheses**.

Idea: to combine pointwise e-values to obtain a functional one.

FUNCTIONAL E-VALUE

Assume now to observe functional data on $T \subset \mathbb{R}$. For $t \in T$, focus on the test of pointwise hypotheses H_0^t against H_1^t, and denote as $E(t)$ the corresponding e-variable and as $e(t)$ the e-value.





FUNCTIONAL E-VALUE

A functional e-value for testing hypotheses

$$H_0 = \bigcap_{t \in T} H_0^t \qquad H_1 = \bigcup_{t \in T} H_1^t$$

can be defined as

$$e = \frac{1}{|T|} \int_T e(t) \mathrm{d}t.$$

It is possible to prove (by Tonelli's theorem) that if $e(t)$ is valid - namely $\mathbb{E}_{H_0^t}[e(t)] \leq 1 \; \forall t$ - then the functional e-value is also valid:

$$\mathbb{E}_{H_0} \left[\frac{1}{|T|} \int_T E(t) \mathrm{d}t \right] \leq 1.$$

PROPERTIES OF FUNCTIONAL E-VALUE

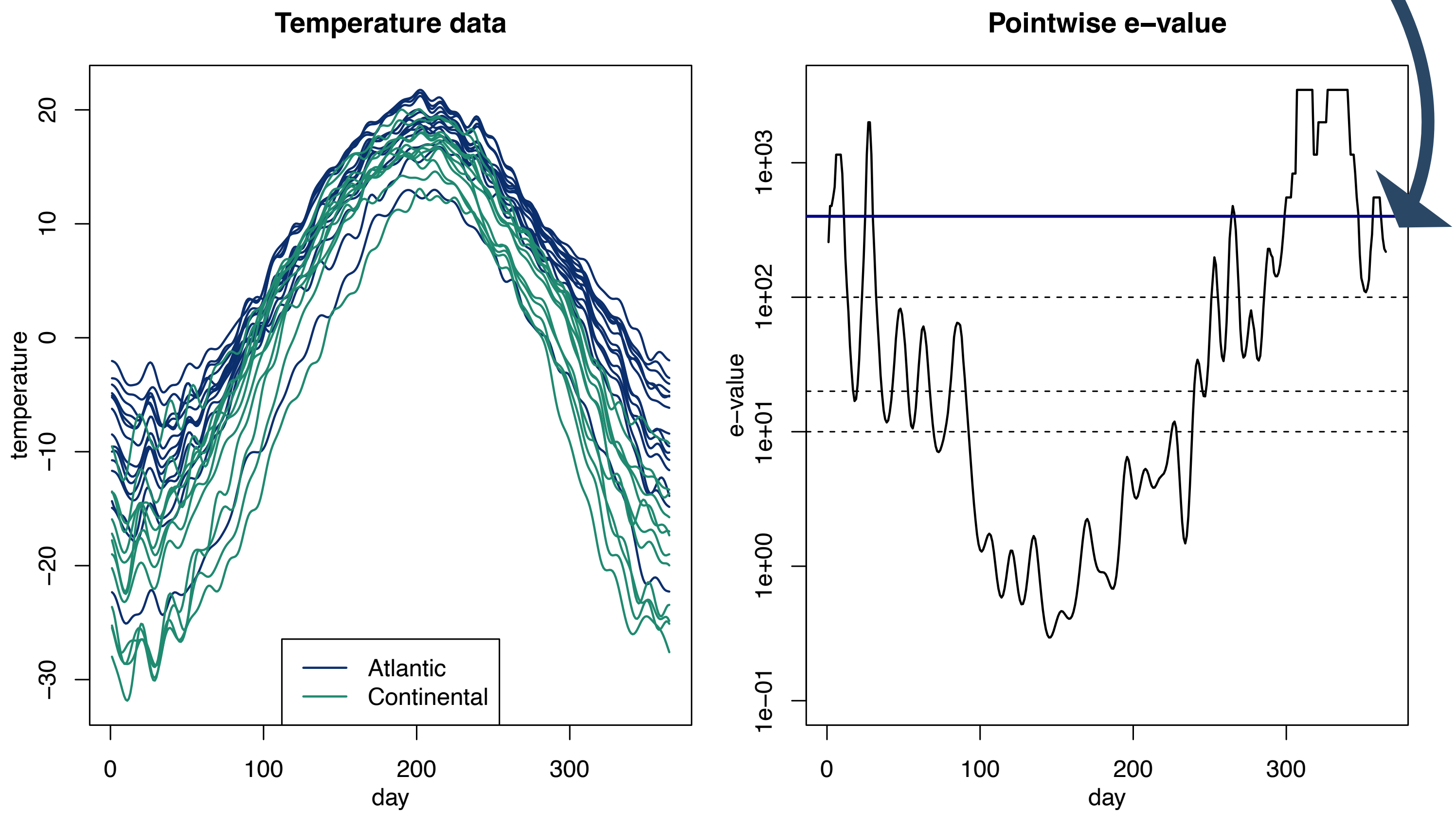
Exactness:
Since the functional e-value e^T is valid, the test that rejects H_0 if $e^T \geq 1/\alpha$ is exact.

Consistency:
If $\lim_{n\to\infty} \mathbb{E}[E^T] = \infty$ and $\lim_{n\to\infty} \mathrm{Var}[E^T] = 0$, the test that rejects H_0 if $e^T \geq 1/\alpha$ is consistent.

- **Exactness** of the resulting test holds without other assumptions and only relies on the validity of pointwise *e*-values.
- For proving **consistency**, one needs to show that E^T goes to infinity in probability. This is not implied directly by the fact that the pointwise *e*-value is consistent.

ANALYSIS OF CANADIAN TEMPERATURE DATA

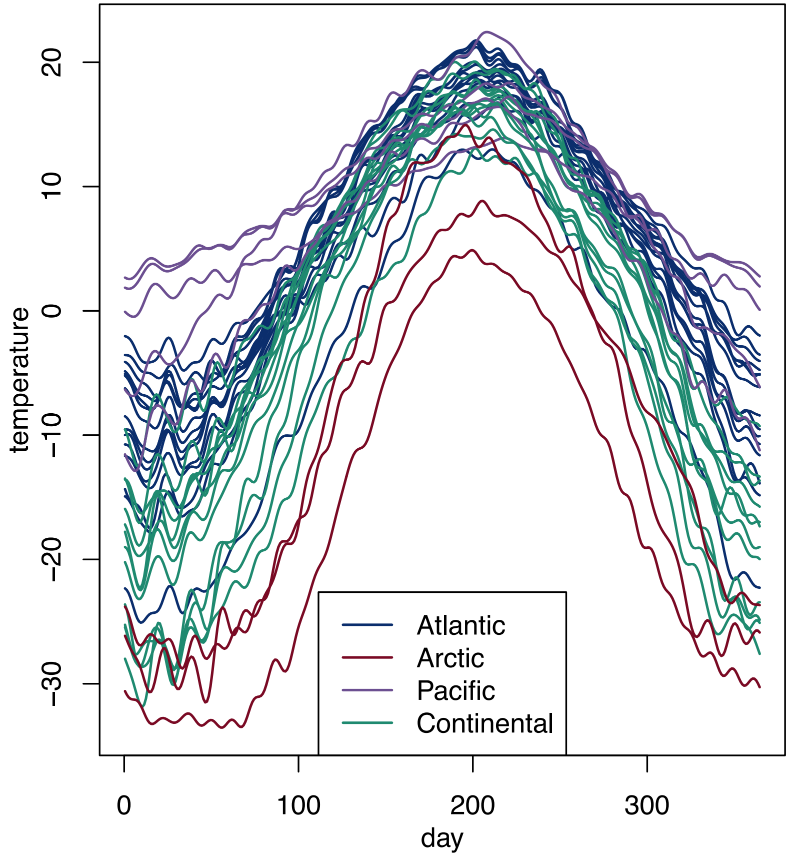
$$e = \frac{1}{|T|} \int_T e(t) \mathrm{d}t$$



ANOVA TEST

In practice, we have a total of four climatic regions.

We are interested in a **functional ANOVA test**.



$$H_0^t : \mathcal{X}_{ij}(t) \stackrel{d}{=} \mathcal{X}_{ij'}(t) \; \forall j, j' \in \{1, \ldots, 4\}$$

$$H_1^t : \exists j, j' \in \{1, \ldots, 4\} : \mathcal{X}_{ij}(t) \stackrel{d}{\neq} \mathcal{X}_{ij'}(t)$$

ANOVA TEST

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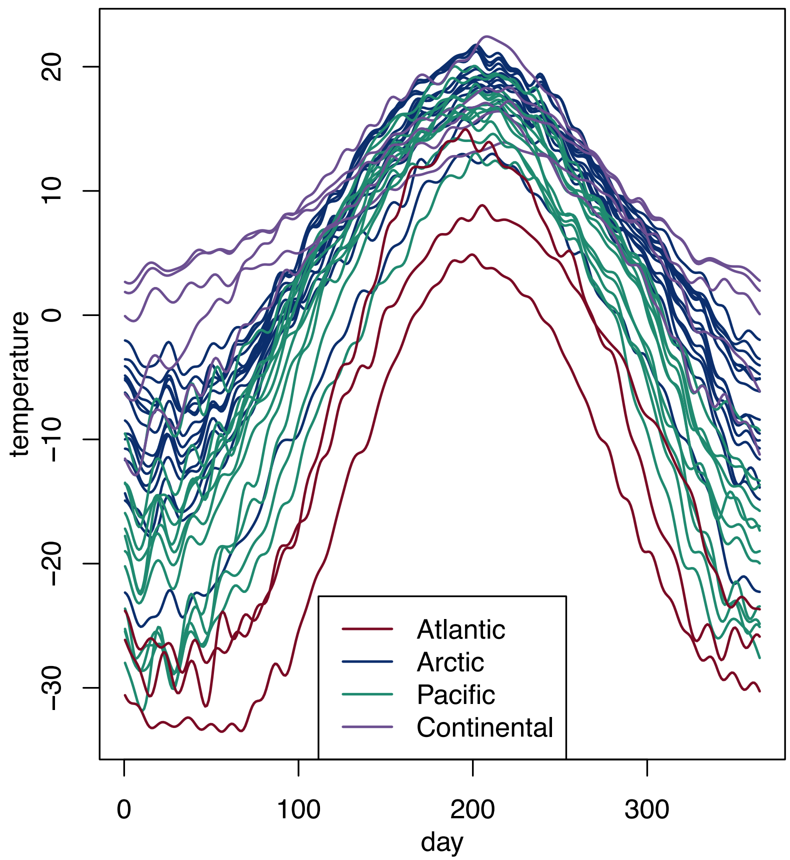
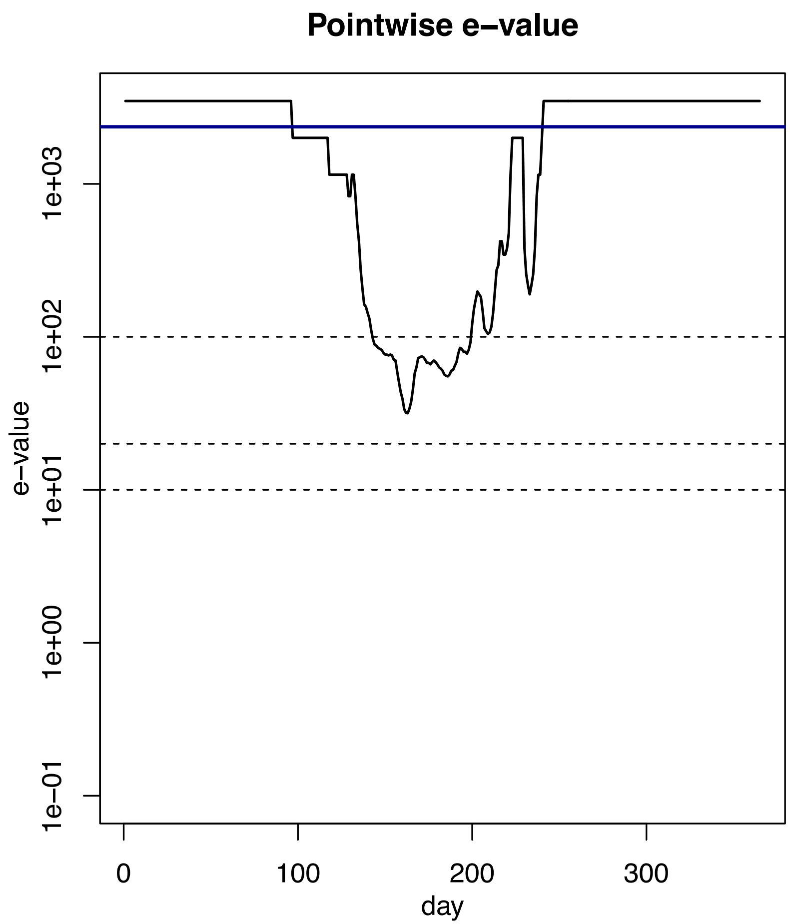




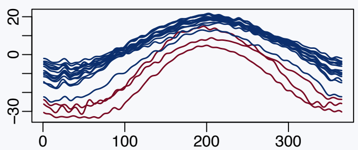
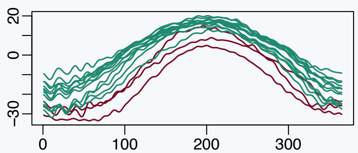
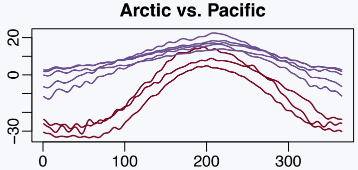
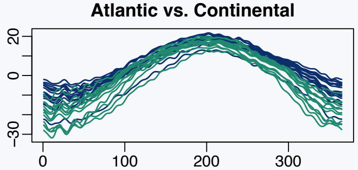
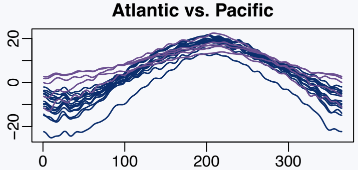
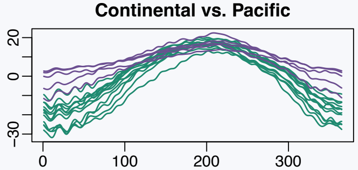
PAIRWISE COMPARISONS WITH MULTIPLICITY ADJUSTMENT

- If we are interested in evaluating all pairwise comparisons, this can also be easily done with *e*-values.
- Since *e*-values can be merged by averaging, multiplicity adjustment on pairwise comparisons after ANOVA can be straightforwardly obtained.
- In particular, we employ an algorithm proposed by Vovk (2020) to adjust e-values (based on an idea similar to the closed testing procedure).
- The adjusted e-values e^* that we obtain are **family-wise valid**, i.e.:

Extended random variables $E_1^*, \ldots, E_K^*$ are family-wise valid if there exist an e-variable E such that

$$\forall k \in \{1, \ldots, K\} \quad E \geq E_k^*.$$

PAIRWISE COMPARISONS WITH MULTIPLICITY ADJUSTMENT

Comparison	E-value	Adjusted e-value
Arctic vs. Atlantic 	64.50	**33.12**
Arctic vs. Continental 	29.33	17.42
Arctic vs. Pacific 	5.53	5.53
Atlantic vs. Continental 	341.06	**96.13**
Atlantic vs. Pacific 	53.35	**29.40**
Continental vs. Pacific 	82.98	**39.28**

DISCUSSION

- Functional *e*-value: a technique to test functional data directly combining pointwise inference.
- To compute a functional *e*-value one only needs to have exact pointwise tests as a starting point. It is much more flexible than existing techniques.
- *e*-values provide a very easy interpretation of the test results.
- It is always possible to convert *e*-values in *p*-values and vice versa.

Future works

Local adjusted *e*-value function to test hypotheses along the domain

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